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Nonlinear Partial Differential Models
Economic Growth & Mean Field Games

Équations aux dérivées partielles non linéaires
Croissance économique et jeux à champ moyen

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Résumé

Ce rapport explore un modèle mathématique découlant d'un problème économique et pouvant être mis sous la forme d'un jeu à champ moyen. Le problème économique considéré est celui de la compétition d'un nombre fini ou infini de joueurs pour du capital et des biens, sous la contrainte qu'ils ne peuvent que connaître la répartition globale en biens et capital des joueurs (d'où le terme de champ moyen). Les joueurs peuvent seulement contrôler leur consommation et leur investissement, et comme les échanges de biens commerciaux peuvent parfois se faire très vite aujourd'hui, les hypothèses de régularité de ces variables de contrôle doivent être très faibles.

On montre dans la deuxième partie du rapport l'existence de trajectoires optimales pour ce modèle sous les seules hypothèses que la consommation et l'investissement soient localement intégrables et que la consommation soit majorée par une fonction intégrable. Pour parler de contrôles définis presque partout, on a besoin de considérer les équations différentielles au sens des distributions et on montre donc une variation du théorème de Cauchy-Lipschitz dans le cadre de fonctions seulement localement intégrables en temps et d'équations définies au sens des distributions.

On étudie aussi une approche numérique de résolution du jeu à champ moyen dans le cas d'un nombre fini de joueurs et l'on soulève les limites de cette approche.

Mots-clés : Équations aux dérivées partielles, jeux à champs moyens, modèle économique, analyse numérique

Abstract

This report explores a mathematical model arising from economy and that can be formulated as a mean field game. The economic situation considered is the one of a group of possibly infinite players in competition for wealth and capital resources. These players can only know the global distribution of other players in the 2D state space of wealth and capital, and can modify their consumption and investment. Today, the exchanges of trade goods can be very fast, what justify the research of results under very weak hypothesis of regularity for the controls.

In the second part of the report, we show the existence of optimal trajectories for this model requiring only locally integrable controls and a consumption dominated by above by an integrable function. In order to have such a result, we define differential equations in the sense of distributions and show a variation of the Cauchy-Lipschitz theorem for functions locally integrable in time.

We also study a numerical approach to solve the mean field model in the case of a finite number of players based on the mean field game interpretation and investigate the limit of such an approach.

Keywords: partial differential equations, mean field games, economic models, numerical analysis

Synthèse du rapport en français

I Présentation du modèle.

Ce rapport étudie un modèle économique présenté et aussi en partie étudié dans [5]. Il met en présence des agents qui ont chacun au temps t une certaine quantité des biens $\mathbf{a}_t$ et de capital $\mathbf{k}_t$ et consomment une quantité $\mathbf{c}_t$ de biens et investissent $\mathbf{p}_t\mathbf{i}_t$ de leurs biens dans du capital, ou $\mathbf{p}_t$ est le prix du capital. Le capital produit des biens et du capital, ce qui définit la fonction de production comptée en terme de valeur en biens, $F(k,p) = \Theta(k,p) + \mathbf{p}_t\Xi(k,p)$. La dépréciation du capital est donnée par $g(k,p)$. Ce modèle conduit donc au système

$$\begin{cases} \dot{\mathbf{a}}_t &= -\mathbf{c}_t - \mathbf{p}_t\mathbf{i}_t + F(\mathbf{k}_t, \mathbf{p}_t) \\ \dot{\mathbf{k}}_t &= \mathbf{i}_t + g(\mathbf{k}_t, \mathbf{p}_t) \end{cases}$$

Chaque agent doit maximiser une fonction de son utilité $u(c,i,a,k)$ donnée ici simplement par $V(\mathbf{a}_0, \mathbf{k}_0, 0)$ où V est la fonction de valeur du problème définie par 1.1,

$$V(a_0, k_0, t) := \sup_{(\mathbf{c},\mathbf{i})\in\mathcal{C}\times\mathcal{I}} \int_t^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s.$$

On peut alors montrer que les solutions de l'équation de Hamilton-Jacobi 1.3,

$$\partial_t V + H(a, k, \mathbf{p}_t, \partial_a V, \partial_k V) = 0,$$

où H est le Hamiltonien du problème (défini par l'équation 1.2) et $V(a,k,T) = 0$, vérifient 1.1.

La particularité des jeux à champs moyens est que chaque agent ne maximise sa trajectoire en ne connaissant que la distribution des autres joueurs (qui peuvent être en nombre infini) $\rho_t(a,k)$. Grâce à l'équation 1.3, on obtient que les trajectoires optimales des joueurs s'écrivent

$$\begin{cases} \dot{\mathbf{a}}_t &= \partial_{q_a} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \partial_a V, \partial_k V) \\ \dot{\mathbf{k}}_t &= \partial_{q_k} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \partial_a V, \partial_k V) \end{cases}$$

et donc que la distribution de joueurs obéit à l'équation de transport 1.4,

$$\partial_t \rho + \partial_a(\partial_{q_a} H\rho) + \partial_k(\partial_{q_k} H\rho) = 0$$

Le Jeux à champ moyen étudié s'écrit donc sous la forme du système 1.6,

$$\begin{cases} \partial_t V &+ & H(a, k, \mathbf{p}_t, \partial_a V, \partial_k V) &= 0 \\ \partial_t \rho &+ & \partial_a(\partial_{q_a} H\rho) + \partial_k(\partial_{q_k} H\rho) &= 0 \end{cases}$$

couplé à une équation de conservation provenant du fait que les agents n'échangent pas avec des agents extérieurs

$$\int_{\mathbb{R}^2} i_t^*(a,k)\rho_t(a,k)\mathrm{d}a\mathrm{d}k = \int_{\mathbb{R}^2} \Xi(k, \mathbf{p}_t)\rho_t(a,k)\mathrm{d}a\mathrm{d}k.$$

II Existence de trajectoires optimales.

1 Théorème d'existence

Dans cette partie, on fixe une fonction de prix et on regarde si il existe toujours une trajectoire optimale pour un joueur avec un certain nombre de biens et de capital au temps $t = 0$. On suppose pour cela que l'utilité est concave et vérifie au plus certaines vitesses de croissance asymptotiquement données par l'hypothèse 2.5, c'est à dire une majorant de la forme

$$C(1 + (c^+)^{\gamma_1} + (a^+)^{\gamma_3} + (k^+)^{\gamma_4} - (c^-)^{\alpha_1} - |i|^{\alpha_2} - (a^-)^{\alpha_3} - (k^-)^{\alpha_4}),$$

où les γ_i sont entre 0 et 1 et les α_i sont supérieurs à 1.

Les résultats principaux sont les suivants:

Théorème d'existence *On pose* $L^1_{R,loc} := \{c \in L^1_{loc}([0,T[), \int_0^T c^+ \leq R\}$ *et* $\mathcal{A}_R := \{c \in L^1_{loc}([0,T[), c \leq TR\}$.
Soit $u \in C^1(\mathbb{R}^4)$ *une fonction concave vérifiant l'hypothèse 2.5 et telle qu'il existe* $C \in \mathbb{R}_+^*$ *tel que*

$$\forall i \in [\![1,4]\!], \left|\frac{\partial u}{\partial x_i}(x_1,...x_4)\right| \leq C\left(1 + \sum_{j=1}^4 |x_j|^{\alpha_j}\right).$$

On suppose que $\mathbf{p} \in L^\infty(0,T)$, *que* F *et* g *sont dominées par une fonction linéaire et que*

- $\forall k \in \mathbb{R}, g(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,
- $\forall k \in \mathbb{R}, F(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,
- *pour tout point de* $\mathbb{R} \times [0,T[$, *il existe un voisinage* $K \times J$ *tel que* $(t \mapsto \sup_K |F(\cdot,t)|) \in L^1(J)$.

Soient $P := \|\mathbf{p}\|_{L^\infty(0,T)}$ *et* $R \in \mathbb{R}$ *tel que* $R \geq \|g(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}P + \|F(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}$ *et*

$$\mathcal{S}(\mathbf{c},\mathbf{i},a_0,k_0) := \begin{cases} \int_0^T u(\mathbf{c}_s,\mathbf{i}_s,\mathbf{a}_s,\mathbf{k}_s)\mathrm{d}s & if\ \tau = T \\ -\infty & if\ \mathbf{k} \underset{\tau}{\to} -\infty\ for\ \tau < T \end{cases}$$

où $\mathbf{a}$, $\mathbf{k}$ *et* τ *sont définis par les problèmes de Cauchy définis par les hypothèses 2.1 et 2.2, c'est à dire le système d'équations différentielles vu à la partie précédentes au sens des distributions. Alors pour tout* $(a_0,k_0) \in \mathbb{R}^2$, *il existe une trajectoire maximisant le problème*

$$\sup_{(\mathbf{c},\mathbf{i}) \in L^1_{R,loc} \times L^1_{loc}([0,T[)} \mathcal{S}(\mathbf{c},\mathbf{i},a_0,k_0),$$

et une trajectoire maximisant le problème

$$\sup_{(\mathbf{c},\mathbf{i}) \in \mathcal{A}_R \times L^1_{loc}([0,T[)} \mathcal{S}(\mathbf{c},\mathbf{i},a_0,k_0).$$

Ce résultat utilise entre autre une variante du théorème de Cauchy-Lipschitz, disponible en annexe (théorème B.2), pour assurer l'existence de trajectoires vérifiant des équations différentielles définies sens des distribution et avec des seconds membres localement intégrables en la seconde variable.

Corollaire (Cas où F et g sont concaves) *Sous les mêmes hypothèses sur* u, $\mathbf{p}$ *et* R, *mais en supposant cette fois que* F *et* g *sont concaves il existe aussi une trajectoire optimale pour toute paire de conditions initiales pour chacun des problèmes.*

2 Conditions pour que le capital reste borné.

On peut par contre toujours être gêné dans notre recherche de solutions numériques à de tels problèmes par le fait que sur l'ensemble de fonctions sur lequel on minimise, il existe des fonctions $\mathbf{k}$ qui ne pourront pas être définies sur $[0, T]$ tout entier mais jusqu'à un réel $\tau < T$, et que même si la solution au problème sera définie sur $[0, T[$, on peut avoir pour la trajectoire optimale $\mathbf{k}_t \underset{t \to T}{\to} -\infty$. La proposition 2.4 indique des hypothèses supplémentaires permettant de se protéger contre l'un ou l'autre de ses problèmes.

3 Optimalité des hypothèses.

Une autre question est celle de l'optimalité des hypothèses. Une justification de l'hypothèse consistant à borner la consommation supérieurement s'obtient en considérant une consommation du type $\mathbf{c}_t := (T - t)^{-\alpha}$ avec $\alpha > 1$ et u vérifiant certaines hypothèses compatibles avec celles des résultats précédents et décrites par la proposition 2.5. On montre en effet alors que $c \in L^1_{loc}([0, T[)$ mais que $\int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s) \mathrm{d}s \underset{t \to T}{\to} +\infty$.

On a aussi une justification du fait que le prix doit être borné dans le cas où il existe au moins une quantité de capital pour laquelle on peut produire montré dans la section 2.3.2, si le prix diverge trop rapidement vers $+\infty$, alors on a aussi $\int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s) \mathrm{d}s \underset{t \to T}{\to} +\infty$.

Ces contre-exemples laissent cependant un petit peu de marge et ne couvrent pas toutes les hypothèses, indiquant qu'il existe peut-être quelques autres combinaisons d'hypothèses possibles sous lesquelles on peut aussi montrer l'existence d'une trajectoire optimale.

III Résultats numériques dans le cas d'un nombre fini de joueurs.

1 Solution de l'équation de transport.

Dans le cas d'un nombre fini $N \in \mathbb{N}^*$ de joueurs, on a une distribution initiale des joueurs de la forme

$$\rho_0 = \frac{1}{N} \sum_{n=1}^{N} \delta_{(a_0^n, k_0^n)},$$

où δ est la mesure de Dirac. On montre alors (proposition 3.1) que la solution est de la forme

$$\rho_t := \frac{1}{N} \sum_{n=1}^{N} \delta_{(\mathbf{a}_t^n, \mathbf{k}_t^n)},$$

ce qui permet de modéliser les joueurs par leurs N trajectoires.

2 Calcul de la solution.

On commence par calculer le Hamiltonien $H(a, k, q_a, q_k)$ à partir de la connaissance de la fonction d'utilité. En revenant à la définition de H et en dérivant la fonction à maximiser, on obtient le système 3.1,

$$\begin{cases} q_a &= \partial_c u(c^*, i^*, a, k) \\ q_k &= p\partial_c u(c^*, i^*, a, k) - \partial_i u(c^*, i^*, a, k) \end{cases}$$

et l'inversion de ce système sous la contrainte $c^* < R$, si elle est possible, donne les contrôles optimaux c^* et i^* et permet donc de calculer le Hamiltonien.

Sous des hypothèses de régularité de V, on peut alors calculer les trajectoires optimales en trouvant la solution du système hamiltonien 3.3 obtenu à partir de l'équation de Hamilton-Jacobi 1.3:

$$\begin{cases} \dot{\mathbf{a}}_t &= \partial_{q_a} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \mathbf{q}_a(t), \mathbf{q}_k(t)) \\ \dot{\mathbf{k}}_t &= \partial_{q_k} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \mathbf{q}_a(t), \mathbf{q}_k(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}_a(t) &= -\partial_a H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \mathbf{q}_a(t), \mathbf{q}_k(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}_k(t) &= -\partial_k H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \mathbf{q}_a(t), \mathbf{q}_k(t)) \end{cases}$$

et muni des conditions de bord $\mathbf{a}_0^n = a_0$, $\mathbf{k}_0 = k_0$, $\mathbf{q}_a(T) = 0$ et $\mathbf{q}_k(T) = 0$.

Tous ces calculs ne peuvent se faire que pour une fonction de prix donnée. Mais le prix doit aussi vérifier l'équation de conservation qui se réécrit

$$\sum_{n=1}^{N} \partial_{q_k} H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) = \sum_{n=1}^{N} g(\mathbf{k}_t^n, \mathbf{p}_t) + \Xi(\mathbf{k}_t^n, \mathbf{p}_t).$$

En résolvant cette équation pour des trajectoires optimales trouvées à partir d'un prix $\mathbf{p}_t$, on obtient un nouveau prix $\tilde{\mathbf{p}}_t$. Il faut alors trouver un point fixe de l'application $\mathbf{p} \mapsto \tilde{\mathbf{p}}$.

3 Résultats numériques.

La partie 3.3 présente des résultats numériques obtenus en appliquant les résultats de la section précédente. Le cas ou le prix n'est pas partout non nul semble poser problème car il peut arriver que l'on obtienne aucune solution au système hamiltonien décrit précédemment.

L'unicité de la solution du jeu à champ moyen de notre modèle est montrée dans [5] mais nécessite une hypothèse d'uniforme continuité sur la fonction u, ce qui est incompatible avec la croissance de u avec certaines de ses variables. Cependant dans le cas d'une fonction u strictement convexe, il semble toujours avoir unicité dans notre cas. Dans le cas d'une fonction qui finit par être stationnaire en la variable c, et donc seulement convexe, on voit parfois apparaître deux trajectoires optimales.

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Presentation of the Host Institution and the Internship Supervisor

This report was written in the context of an internship of six months in the King Abdullah University for Science & Technology (KAUST) in the city of Thuwal in Saudi Arabia, near the city of Jeddah. KAUST is a university created in September 2009 aspiring to regroup international students and researchers. It has three main divisions, the Biological and Environmental Science and Engineering Division (BESE), the Physical Science and Engineering Division (PSE) and the Computer, Electrical and Mathematical Science and Engineering Division (CEMSE).

I worked in one of the three main departments of the CEMSE division called Applied Mathematics & Computational Sciences (AMCS), in the group of the professor Diogo Gomes that studies a particular system of partial differential equations called mean field games.

Introduction

Mean field games are a class of partial derivative equations describing the behaviour of a distribution of players that are optimizing a functional knowing only the global distribution of the other players. These players can be for example human agents trying to minimize a cost or particles minimizing a potential. They are characterized by a state variable, like wealth or position, and a control variable, like consumption or speed. The mathematical problem is written as a system of two coupled partial derivative equations: a Hamilton-Jacobi equation linked to the optimization problem of each player and a transport equation describing the dynamic of the density of players. The particularity of mean field games is the possibility of considering an infinite number of players. Mean field games can indeed be seen as a generalization of some models of game theory when the number of players becomes infinite (see for example [4]).

Mean field games were first introduced by Pierre Louis Lions and Jean Michel Lasry (see [14], [11], [12] and [13]) and were also studied independently at the same time by Peter Caines, Minyi Huang, and Roland Malhamé (see [10] and [9]). Since then, several new aspects, generalizations and applications have been studied. Mathematical surveys can be found in [2] and [6] and several applications can be found in [8].

In the current report, we study a simple economic model about a wealth and capital accumulation problem. This problem and other related economic problems that can be studied with a mean field games approach are presented in the book by Diogo A. Gomes, Levon R. Nurbekyan and Edgard A. Pimentel (see [5]). In this book, some mathematical results about the model studied in this report are shown, as the uniqueness of the solution in the case of a uniformly convex utility function and the behaviour of the distribution of players in the N-agent approximation case.

The first part of this report briefly presents the model and its mean field game formulation as defined in [5], the second chapter study the hypotheses of well definiteness of the model in the case of a utility function of given asymptotic behaviour and the third chapter shows a possible numerical approach and some numerical results in the case of a finite number of agents.

Notations And Definitions

neighbourhood of x	open subset containing x.
near x	in a neighbourhood of x.
$\mathbb{N}$	set of integers.
$[\![n, m]\!]$	$[n, m] \cap \mathbb{N}$.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}^*$	$\mathbb{R}\backslash\{0\}$.
E	a vector space.
$\bar{B}(x, r)$	$\{x \in E, \|x\| \leq r\}$.
$\mathcal{B}(A)$	space of Lebesgue measurable functions from A to $\mathbb{R}$.
μ_L	Lebesgue measure.
$\int f(x)\mu(\mathrm{d}x)$	integral of f with respect to the measure μ.
$\int \mu$	$\int \mu(\mathrm{d}x)$ where μ is a measure.
$\int f = \int f(x)\mathrm{d}x$	$\int f(x)\mu_L(\mathrm{d}x)$ where f is a Lebesgue measurable function.
$\mathcal{L}^p(A)$	$\{u \in \mathcal{B}(A), \int_A \|u\|^p < +\infty\}$.
$\mathcal{L}^\infty(A)$	$\{u \in \mathcal{B}(A), u$ is bounded on $A\}$.
$u \overset{a.e.}{=} v$	$\overset{\text{def}}{\Leftrightarrow} \mu_L(\{u = v\}) = 0$.
$L^p(E)$	$\mathcal{L}^p(E)/ \overset{a.e.}{=}$ (Lebesgue spaces).
$L^p_{loc}(E)$	$\{u \in \mathcal{B}(E), \forall K \subset E$ compact, $u_{\|K} \in \mathcal{L}^p(K)\}/ \overset{a.e.}{=}$.
$W^{n,p}(E)$	Sobolev spaces.
$\mathbf{D}^n(E, F)$	is the space of n times differentiable applications from E to F.
$C^n(E, F)$	applications from $\mathbf{D}^n(E, F)$ such that the n-th derivative is continuous.
$C^n(E)$	$C^n(E, \mathbb{R})$.
$\mathcal{D}(E)$	C^∞ functions with compact support.
$\mathcal{D}'(E)$	space of distributions on E.
$\rightharpoonup$	weak convergence.
u^+	$\max(u, 0)$.
u^-	$u^+ - u$.
u_t	value of u at time t.
$\dot{u}_t$	value of the derivative of u at time t.

Chapter 1

Presentation of the Model

1.1 Microeconomic Settings

1.1.1 Price and Goods Dynamics

In the model we study in this report, presented in [5], we consider a set of agents that can produce and trade capital and goods. Each agent at time t has a certain amount of capital $\mathbf{k}_t$ and goods $\mathbf{a}_t$, and can decide to consume a quantity $\mathbf{c}_t$ of goods and to exchange an amount $\mathbf{e}_t$ of capital against goods at a price $\mathbf{p}_t$ with other agents.

The capital produces a certain amount of goods, given by a function $\Theta(k, p)$, and a certain amount of capital, $\Xi(k, p)$. There is also a depreciation of the capital. It is given by a function $g(k, p)$. So the capital and goods of a given agent are driven by the following system:

$$\begin{cases} \dot{\mathbf{a}}_t &= -\mathbf{c}_t - \mathbf{p}_t\mathbf{e}_t + \Theta(\mathbf{k}_t, \mathbf{p}_t) \\ \dot{\mathbf{k}}_t &= g(\mathbf{k}_t, \mathbf{p}_t) + \mathbf{e}_t + \Xi(\mathbf{k}_t, \mathbf{p}_t) \end{cases}$$

Equivalently, we can say that an agent chooses the investment $\mathbf{i}_t = \mathbf{e}_t + \Xi(\mathbf{k}_t, \mathbf{p}_t)$. Then by defining the global production function $F(k, p) := \Theta(\mathbf{k}_t, \mathbf{p}_t) + \mathbf{p}_t\Xi(\mathbf{k}_t, \mathbf{p}_t)$ we obtain the system

$$\begin{cases} \dot{\mathbf{a}}_t &= -\mathbf{c}_t - \mathbf{p}_t\mathbf{i}_t + F(\mathbf{k}_t, \mathbf{p}_t) \\ \dot{\mathbf{k}}_t &= \mathbf{i}_t + g(\mathbf{k}_t, \mathbf{p}_t) \end{cases}$$

1.1.2 Agents Individual Objective

Each agent has preferences on the value of his consumption, investment, amount of goods and amount of capital, given by a utility function $u(c, i, a, k)$. A usual assumption about such a utility function is the fact that it is increasing and concave in each variable. It comes from the fact that, for example considering the quantity of goods of an agent, he will always choose the bigger amount of goods if he has the choice between two amounts, and the fact that when he has a lot of goods, adding a little quantity will no change a lot his satisfaction.

So, the goal of an agent will be to maximize his utility during a given time T for a given initial amount of goods $\mathbf{a}_0 = a_0$ and capital $\mathbf{k}_0 = k_0$. We define the value function as

$$V(a_0, k_0, t) := \sup_{(\mathbf{c},\mathbf{i})\in\mathcal{C}\times\mathcal{I}} \int_t^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s, \tag{1.1}$$

where $\mathcal{C}$ and $\mathcal{I}$ are the functional spaces on $(0, T)$ where the controls $\mathbf{i}$ and $\mathbf{c}$ can be chosen. As it is shown in [5], by defining the Hamiltonian

$$H(a, k, p, q_a, q_k) := \sup_{(c,i)\in\mathbb{R}^2} ((-c - pi + F(k, p))q_a + (i + g(k, p))q_k + u(c, i, a, k)), \tag{1.2}$$

we obtain that if $v \in C^1(\mathbb{R}^2 \times [0,T])$ is solution of the Hamilton-Jacobi equation

$$\partial_t v(a,k,t) + H(a,k,\mathbf{p}_t, \partial_a v(a,k,t), \partial_k v(a,k,t)) = 0, \tag{1.3}$$

with $v(a,k,T) = 0$, then v is the value function. So the optimal controls $c_t^*(a,k)$ and $i_t^*(a,k)$ are given by

$$\begin{array}{rcl} \partial_{q_a} H(a,k,\mathbf{p}_t,\partial_a V, \partial_k V) & = & -c_t^* - \mathbf{p}_t i_t^* + F(k,\mathbf{p}_t) \\ \partial_{q_k} H(a,k,\mathbf{p}_t,\partial_a V, \partial_k V) & = & i_t^* + g(k,\mathbf{p}_t) \end{array}$$

1.2 Macroeconomic Settings

On the macroeconomic scale, the agents are described by a density of agents ρ, where $\rho_t(a,k)$ indicates the proportion of agents that have $\mathbf{a}_t = a$ and $\mathbf{k}_t = k$ and for all $t \in [0,T]$,

$$\int_{\mathbb{R}^2} \rho_t(a,k) \mathrm{d}a \mathrm{d}k = 1.$$

As the agents follow the system

$$\left\{ \begin{array}{rcl} \dot{\mathbf{a}}_t & = & \partial_{q_a} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \partial_a V(\mathbf{a}_t,\mathbf{k}_t,t), \partial_k V(\mathbf{a}_t,\mathbf{k}_t,t)) \\ \dot{\mathbf{k}}_t & = & \partial_{q_k} H(\mathbf{a}_t, \mathbf{k}_t, \mathbf{p}_t, \partial_a V(\mathbf{a}_t,\mathbf{k}_t,t), \partial_k V(\mathbf{a}_t,\mathbf{k}_t,t)) \end{array} \right.$$

the repartition of players follow the transport equation

$$\partial_t \rho + \partial_a(\partial_{q_a} H \rho) + \partial_k(\partial_{q_k} H \rho) = 0 \tag{1.4}$$

As the agents can only exchange goods and capital with other agents, we can write a conservation equation for $t \in [0,T]$: $\int_{\mathbb{R}^2} \mathbf{e}_t^*(a,k)\rho(a,k,t)\mathrm{d}a\mathrm{d}k = 0$, or equivalently

$$\int_{\mathbb{R}^2} i_t^*(a,k)\rho_t(a,k)\mathrm{d}a\mathrm{d}k = \int_{\mathbb{R}^2} \Xi(k,\mathbf{p}_t)\rho_t(a,k)\mathrm{d}a\mathrm{d}k \tag{1.5}$$

which determines $\mathbf{p}_t$.

1.3 Mean Field Game Formulation of the problem

The Hamilton-Jacobi equation (1.3) and the transport equation (1.4) together bring a mean-field game formulation of the problem

$$\left\{ \begin{array}{rcccl} \partial_t V & + & H(a,k,\mathbf{p}_t,\partial_a V,\partial_k V) & = & 0 \\ \partial_t \rho & + & \partial_a(\partial_{q_a} H\rho) + \partial_k(\partial_{q_k} H\rho) & = & 0 \end{array} \right. \tag{1.6}$$

with $V(a,k,T) = 0$, $\rho_0(a,k)$ is given and $\mathbf{p}_t$ is such that the equilibrium equation (1.5) is satisfied.

Chapter 2

Existence of Optimal Trajectories

Let $T \in \mathbb{R}_+^*$. In this chapter, we are trying to find under what assumptions there exist a solution to the maximization problem that defines the value function,

$$\sup_{(\mathbf{c},\mathbf{i})\in\mathcal{C}\times\mathcal{I}} \int_t^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s, \tag{2.1}$$

where $\mathbf{a}$ and $\mathbf{k}$ are defined as in 1.1. So, we are trying to know if there exist optimal controls for a player of given initial quantity of goods and capital.

Assumption 2.1 (Definition of the capital)**.** $\mathbf{k}$ *is the maximal continuous solution on* $[0, T]$ *to*

$$\begin{cases} \dot{\mathbf{k}}_t &= \mathbf{i}_t + g(\mathbf{k}_t, \mathbf{p}_t) \text{ in } \mathcal{D}'(0,T) \\ \mathbf{k}_0 &= k_0 \text{ is given,} \end{cases}$$

where $\mathbf{i}$ *and* $\mathbf{p}$ *are defined on* $[0, T]$*.*
Let $\tau := \sup(t \in [0, T], \mathbf{k}$ *is defined on* $[0, \tau[)$*.*

Assumption 2.2 (Definition of the goods)**.** $\mathbf{a}$ *is the maximal continuous solution on* $[0, T]$ *to*

$$\begin{cases} \dot{\mathbf{a}}_t &= -\mathbf{c}_t - \mathbf{p}_t\mathbf{i}_t + F(\mathbf{k}_t, \mathbf{p}_t) \text{ in } \mathcal{D}'(0,T) \\ \mathbf{a}_0 &= a_0 \text{ is given,} \end{cases}$$

where $\mathbf{c}, \mathbf{i}$ *and* $\mathbf{p}$ *are defined on* $[0, T]$ *and* $\mathbf{k}$ *is defined on* $[0, T]$ *or on* $[0, \tau[$ *for a given* $\tau \in]0, T[$*.*

Assumption 2.3 (g is sub-linear)**.** *There exists* $(g_1, g_2) \in \mathcal{L}^\infty_{loc}(\mathbb{R})^2$ *such that*

$$\forall (k, p) \in \mathbb{R}^2, g(k, p) \leq g_1(p)k + g_2(p).$$

Assumption 2.4 (F is sub-linear)**.** *There exists* $(F_1, F_2) \in \mathcal{L}^\infty_{loc}(\mathbb{R})^2$ *such that*

$$\forall (k, p) \in \mathbb{R}^2, F(k, p) \leq F_1(p)k + F_2(p).$$

Assumption 2.5. *There exists* $(\gamma_j)_{j\in[\![1,4]\!]} \in [0, 1]^4$*,* $(\alpha_j)_{j\in[\![1,4]\!]} \in [1, +\infty[^4$ *and* $C_u \in \mathbb{R}_+^*$ *such that*

$$\forall (c, i, a, k) \in \mathbb{R}^4, u(c, i, a, k) \leq C_u(1+(c^+)^{\gamma_1}+(a^+)^{\gamma_3}+(k^+)^{\gamma_4}-(c^-)^{\alpha_1}-|i|^{\alpha_2}-(a^-)^{\alpha_3}-(k^-)^{\alpha_4}).$$

Assumption 2.6. *There exists* $(g_3, g_4, \alpha) \in \mathcal{L}^\infty_{loc}(\mathbb{R})^2\times]1, +\infty[$ *such that*

$$\forall (k, p) \in \mathbb{R}_- \times \mathbb{R}, g(k, p) \geq g_3(p)|k|^\alpha + g_4(p).$$

Definition 2.1. *Let A, B and C be normed vector spaces, $\Omega \times I \subset A \times B$ be open subsets. $F : \Omega \times I \to C$ is* ***locally uniformly Lipschitz*** *in the first variable* $\overset{\text{def}}{\Leftrightarrow}$

$$\exists L > 0, \forall (x_0, t_0) \in \Omega \times I, \|F(x_1, t) - F(x_2, t)\| \leq L\|x_1 - x_2\| \textit{ near } (x_0, t_0).$$

Definition 2.2. *Let A, B and C be normed vector spaces, $\Omega \times I \subset A \times B$ be open subsets. $F : \Omega \times I \to C$ is* ***locally uniformly integrable*** *in the second variable* $\overset{\text{def}}{\Leftrightarrow}$

$$\forall (x_0, t_0) \in \Omega \times I, \exists X \times J \text{ neighbourhood of } (x_0, t_0), \left(t \mapsto \sup_{x \in X} |F(x, t)| \right) \in L^1(J).$$

2.1 Existence of a Maximizer

2.1.1 Existence for c and i Integrable

Lemma 2.1.1 (Bound on $\mathbf{k}^+$)**.** *Let* $\mathbf{k}$ *and* τ *be defined by assumption 2.1 with* $\mathbf{i} \in L^1(0,T)$ *and assume there exists* $(G_1, G_2) \in \mathbb{R}^2$ *such that for a.e.* $t \in [0,\tau]$, $\forall k \in \mathbb{R}, g(k,\mathbf{p}_t) \leq G_1 k + G_2$.

Then $\exists M \in \mathbb{R}, \forall t \in [0,\tau[, \mathbf{k}_t \leq M$, *where* M *depends only on* $\mathbf{i}, k_0, G_1, G_2$ *and* τ.

Proof

As $\dot{\mathbf{k}}_t \leq \mathbf{i}_t + G_1\mathbf{k}_t + G_2$ a.e., by applying Gronwall's Inequality (Proposition A.2) for $t \in [0,\tau[$, we obtain

$$\begin{aligned}
\mathbf{k}_t &\leq e^{G_1 t}\left(k_0 + \textstyle\int_0^t (\mathbf{i}_s + G_2)e^{-G_1 s}\mathrm{d}s\right)\\
&\leq e^{G_1 t}\left(k_0 + G_2 G_1^{-1}(1-e^{-G_1 t}) + \textstyle\int_0^t |\mathbf{i}_s| e^{-G_1 s}\mathrm{d}s\right)\\
&\leq e^{G_1 \tau}\left(|k_0| + |G_2 G_1^{-1}|(1+e^{|G_1|\tau}) + \|\mathbf{i}\|_{L^1(0,T)}e^{|G_1|\tau}\right).
\end{aligned}$$

□

Lemma 2.1.2 (Local Definiteness of $\mathbf{a}$)**.** *Let* $(\mathbf{c},\mathbf{i}) \in L^1_{loc}([0,T[)^2$, $\mathbf{p} \in L^\infty(0,T)$, $F : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $(k,t) \mapsto F(k,\mathbf{p}_t)$ *is locally uniformly integrable in the second variable and* $\mathbf{k} \in C^0[0,\tau[$ *with* $\tau \in]0,T]$.

Then there exists $\mathbf{a} \in C^0[0,\tau[$ *absolutely continuous satisfying assumption 2.2.*

If $(\mathbf{k},\mathbf{c},\mathbf{i}) \in C^0[0,T[\times L^1(0,T)^2$, $\mathbf{k}^+$ *is bounded and* F *is increasing in the first variable and such that* $\forall k \in \mathbb{R}, F(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$, *then* $\mathbf{a} \in C^0[0,T[$ *and* $\mathbf{a}^+$ *is bounded.*

Proof

Let $t_0 \in]0,\tau[$, $K := \|\mathbf{k}\|_{C^0([0,t_0])}$ and $P := \|\mathbf{p}\|_{L^\infty(0,T)}$. As $(k,t) \mapsto F(k,\mathbf{p}_t)$ is locally uniformly integrable in the second variable, $\mathcal{F} := (t \mapsto \|F(\cdot,\mathbf{p}_t)\|_{\mathcal{L}^\infty(-K,K)}) \in L^1(0,t_0)$ and we have

$$\int_0^{t_0} |-\mathbf{c}_s - \mathbf{p}_s\mathbf{i}_s + F(\mathbf{k}_s,\mathbf{p}_s)|\mathrm{d}s \leq \|\mathbf{c}\|_{L^1(0,t_0)} + P\|\mathbf{i}\|_{L^1(0,t_0)} + \|\mathcal{F}\|_{L^1(0,t_0)}.$$

So, by theorem B.2, $\mathbf{a}$ is well defined on $[0,\tau[$ and is absolutely continuous.

If $(\mathbf{k},\mathbf{c},\mathbf{i}) \in C^0[0,T[\times L^1(0,T)^2$, there exists $M \in \mathbb{R}_+$ such that $\mathbf{k} < M$ and F is increasing in the first variable, then we have

$$\begin{aligned}
\mathbf{a}_t &= a_0 + \int_0^t \dot{\mathbf{a}}\\
&\leq |a_0| + \int_0^t |-\mathbf{c}_s - \mathbf{p}_s\mathbf{i}_s| + F(M,\mathbf{p}_s)\mathrm{d}s\\
&\leq \|\mathbf{c}\|_{L^1(0,T)} + P\|\mathbf{i}\|_{L^1(0,T)} + T\|F(M,\cdot)\|_{\mathcal{L}^\infty(-P,P)}.
\end{aligned}$$

□

Lemma 2.1.3 (The Constant Solution)**.** *Let* $(a_0,k_0,T) \in \mathbb{R}^3$, $u \in C^0(\mathbb{R}^4), \mathbf{p} \in L^\infty(0,T)$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$ *and* $F : \mathbb{R}^2 \to \mathbb{R}$ *be sub-linear (assumptions 2.3 and 2.4), and be such that*

- $\forall k \in \mathbb{R}, g(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,
- $\forall k \in \mathbb{R}, F(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$.

Define $\forall t \in [0,T], \bar{\mathbf{c}}_t := g(k_0, \mathbf{p}_t)\mathbf{p}_t + F(k_0, \mathbf{p}_t)$ *and* $\bar{\mathbf{i}}_t = -g(k_0, \mathbf{p}_t)$.

Then the constant functions $\bar{\mathbf{k}} : t \mapsto k_0$ *and* $\bar{\mathbf{a}} : t \mapsto a_0$ *satisfy the Cauchy problems from assumptions 2.2 and 2.1. Moreover,* $U_0 := \int_0^T |u(\bar{\mathbf{c}}_s, \bar{\mathbf{i}}_s, \bar{\mathbf{a}}_s, \bar{\mathbf{k}}_s)| \mathrm{d}s$ *is finite.*

Proof

As $\mathbf{p} \in L^\infty(0,T)$ and $g(k_0,\cdot)$ and $F(k_0,\cdot)$ are bounded on $[-\|\mathbf{p}\|_{L^\infty(0,T)}, \|\mathbf{p}\|_{L^\infty(0,T)}]$, we obtain that $(\bar{\mathbf{c}}, \bar{\mathbf{i}}) \in L^\infty(0,T)^2$. As $u \in C^0(\mathbb{R}^4)$ and $\bar{\mathbf{c}}, \bar{\mathbf{i}}, \bar{\mathbf{a}}$ and $\bar{\mathbf{k}}$ are in $L^\infty(0,T)$, then $t \mapsto u(\bar{\mathbf{c}}_t, \bar{\mathbf{i}}_t, \bar{\mathbf{a}}_t, \bar{\mathbf{k}}_t) \in L^\infty(0,T)$ and $\int_0^T u(\bar{\mathbf{c}}_s, \bar{\mathbf{i}}_s, \bar{\mathbf{a}}_s, \bar{\mathbf{k}}_s)\mathrm{d}s$ is finite. □

Proposition 2.1 (Uniform Bounds on the Trajectories)**.** *Let* $F : \mathbb{R}^2 \to \mathbb{R}$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$ *be sub-linear (assumptions 2.3 and 2.4),* $(\mathbf{c}, \mathbf{i}) \in L^1(0,T)^2$ *and* $(\mathbf{a}, \mathbf{k}) \in C^0([0,T[)^2$ *absolutely continuous be defined by assumptions 2.1 and 2.2, be such that* $(\mathbf{k}^+, \mathbf{a}^+) \in L^{\gamma_3}(0,T) \times L^{\gamma_4}(0,T)$ *and such that there exists* $U_0 \in \mathbb{R}_+$ *such that* $\int_0^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s > -U_0$*. Let* u *verify assumption 2.5 and* $\mathbf{p} \in L^\infty(0,T)$*.*

Then, there exists a constant $C \in \mathbb{R}_+$*, depending only on* $F, g, |k_0|, \mathbf{p}, T, \gamma_1, \gamma_3, \gamma_4, C_u, U_0$ *and* $R := \|\mathbf{c}^+\|_{L^1(0,T)}$*, such that*

$$\mathbf{a}^+ \leq C, \mathbf{k}^+ \leq C,$$

$$\int_0^T (\mathbf{c}^-)^{\alpha_1} \leq C,\ \int_0^T |\mathbf{i}|^{\alpha_2} \leq C,\ \int_0^T (\mathbf{a}^-)^{\alpha_3} \leq C,\ \int_0^T (\mathbf{k}^-)^{\alpha_4} \leq C,$$

and also such that

$$\int_0^T |\mathbf{c}| \leq C,\ \int_0^T |\mathbf{a}|^{\alpha_3} \leq C,\ \int_0^T |\mathbf{k}|^{\alpha_4} \leq C.$$

Proof

Define $A : t \mapsto \int_0^t (\mathbf{a}^+)^{\gamma_3}$, $K : t \mapsto \int_0^t (\mathbf{k}^+)^{\gamma_4}$, $I : t \mapsto \int_0^t |\mathbf{i}|$ and $U_R := 2T + R + U_0 C_u^{-1} \in \mathbb{R}_+$.

(i) There exists $(C_{2,R}, C_2) \in \mathbb{R}_+^2, \mathbf{k}_t \leq C_{2,R} + C_2 A_t$**.**

As $\mathbf{p} \in L^\infty(0,T)$, by sub-linearity of g (assumption 2.3), there exists $(G_0, G_1) \in \mathbb{R}^2$ such that for a.e. $t \in [0,T[, g(\mathbf{k}_t, \mathbf{p}_t) \leq G_0 \mathbf{k}_t + G_1$.

Let $t \in [0,T[$. $\dot{\mathbf{k}}_t = \mathbf{i}_t + g(\mathbf{k}_t, \mathbf{p}_t) \leq \mathbf{i}_t + G_0\mathbf{k}_t + G_1$. Thus, by the Gronwall's Inequality (Proposition A.2) we obtain

$$\begin{aligned} \mathbf{k}_t &\leq e^{G_0 t}\left(k_0 + \int_0^t (\mathbf{i}_s + G_1)e^{-G_0 s}\mathrm{d}s\right) \\ &\leq e^{G_0 t}\left(|k_0| + \int_0^t (|\mathbf{i}_s| + |G_1|)e^{|G_0|s}\mathrm{d}s\right) \\ &\leq C_0 + \widetilde{C_0} I_t. \end{aligned}$$

where $\widetilde{C_0} = e^{|G_0|T}$ and $C_0 = \widetilde{C_0}\left(|k_0| + |G_1|T\widetilde{C_0}\right)$.

As u verifies assumption 2.5, we obtain for a.e. $t \in [0,T]$

$$|\mathbf{i}_t| \leq 1 + |\mathbf{i}_t|^{\alpha_2} \leq 2 + \mathbf{c}^+ + (\mathbf{a}^+)^{\gamma_3} + (\mathbf{k}^+)^{\gamma_4} - u(\mathbf{c}_t, \mathbf{i}_t, \mathbf{a}_t, \mathbf{k}_t)C_u^{-1}$$

where we used Lemma C.1.3 twice. By integrating in time, we get

$$I_T \leq U_R + A_T + \int_0^T (C_0 + \widetilde{C_0} I_s)^{\gamma_4}\mathrm{d}s.$$

As $1-\gamma_4 > 0$, there exists $m \in \mathbb{R}_+$ such that $\dfrac{\widetilde{C_0}T}{m^{1-\gamma_4}} \leq 1$. Thus, using Lemma C.1.3 with $\alpha = 1$, we have

$$\begin{aligned} I_T &\leq U_R + A_T + \left(m^{\gamma_4} + \frac{C_0}{m^{1-\gamma_4}}\right)T + \frac{\widetilde{C_0}}{m^{1-\gamma_4}}\int_0^T I_s \mathrm{d} \\ &\leq U_R + A_T + \left(m^{\gamma_4} + \frac{C_0}{m^{1-\gamma_4}}\right)T + \frac{\widetilde{C_0}T}{m^{1-\gamma_4}} I_T. \end{aligned}$$

Thus, we have

$$I_T \leq C_1 A_T + C_{1,R},$$

where $C_1 = \dfrac{m^{1-\gamma_4}}{m^{1-\gamma_4} - \widetilde{C_0}T}$ and $C_{1,R} = (U_R + (m^{\gamma_4} + C_0 m^{\gamma_4 - 1})T)C_1$.

Therefore, we obtain $\forall t \in [0,T[, \mathbf{k}_t \leq C_0 + \widetilde{C_0}C_{1,R} + \widetilde{C_0}C_1 A_T$.

(ii) There exists $(C_{3,R}, C_3) \in \mathbb{R}_+^2, \int_0^t \mathbf{k}^- \leq C_{3,R} + C_3 A_T$ and $\int_0^t \mathbf{c}^- \leq C_{3,R} + C_3 A_T$.

As u verifies assumption 2.5, we have

$$\int_0^t \mathbf{k}^- \leq U_R + A_T + \int_0^T (C_{2,R} + C_2 A_s)^{\gamma_4} \mathrm{d}s.$$

Thus, using Lemma C.1.2 for $\alpha = 1$, we have

$$\int_0^t \mathbf{k}^- \leq T + A_T + U_R + (1 + C_{2,R})T + C_2 T A_T.$$

Thus we have the result for $C_3 := 1 + C_2 T$ and $C_{3,R} := (2 + C_{2,R})T + U_R$.
The same demonstration holds for $\mathbf{c}^-$, by replacing α_4 by α_1.

(iii) There exists $(C_{4,R}, C_4) \in \mathbb{R}_+^2, \mathbf{a}_t^+ \leq C_{4,R} + C_4 A_T$.

As for g, the sub-linearity of F (assumption 2.4), gives two constants $(F_0, F_1) \in \mathbb{R}^2$ such that for a.e. $t \in [0,T[, F(\mathbf{k}_t, \mathbf{p}_t) \leq F_0 \mathbf{k}_t + F_1$.

$$\begin{aligned} \mathbf{a}_t &= a_0 - \int_0^t (\mathbf{c} + \mathbf{p}\mathbf{i}) + \int_0^t F(\mathbf{k}_s, \mathbf{p}_s)\mathrm{d}s \\ &\leq |a_0| + \int_0^t \mathbf{c}^- + R + \|\mathbf{p}\|_{L^\infty(0,T)} \int_0^t |\mathbf{i}| + F_1 t + F_0 \int_0^t \mathbf{k} \\ &\leq |a_0| + C_{3,R} + C_3 A_T + R + \|\mathbf{p}\|_{L^\infty}(1 + C_1 A_T + C_{1,R}) + (F_1 + F_0(C_{2,R} + C_2 A_T))T \\ &\leq C_{4,R} + C_4 A_T, \end{aligned}$$

where $C_4 = C_3 + \|\mathbf{p}\|_{L^\infty(0,T)} C_1 + F_0 C_2 T$ and $C_{4,R} = |a_0| + C_{3,R} + R + \|\mathbf{p}\|_{L^\infty(0,T)}(1 + C_{1,R}) + (F_1 + F_0 C_{2,R})T$. Therefore, as $C_{4,R}, C_4$ and A_T are positive, $\mathbf{a}_t^+ \leq C_{4,R} + C_4 A_T$.

(iv) Bounds of $\mathbf{a}^+$ and $\mathbf{k}^+$

As $1-\gamma_3 > 0$, there exists $n \in \mathbb{R}_+$ such that $\dfrac{C_4 T}{n^{1-\gamma_3}} \leq 1$. Thus, by using Lemma C.1.3 with $\alpha = 1$, we have

$$A_T = \int_0^T (\mathbf{a}^+)^{\gamma_3} \leq \left(n^{\gamma_3} + C_{4,R} n^{\gamma_3 - 1} + \frac{C_4}{n^{1-\gamma_3}} A_T\right) T.$$

Thus, we have

$$A_T \leq \frac{T}{n^{1-\gamma_3} - C_4 T}(n + C_{4,R}).$$

Then, (i) and (iii) give the bounds of $\mathbf{a}^+$ and $\mathbf{k}^+$.

The bounds on $\mathbf{c}^-, |\mathbf{i}|, \mathbf{a}^-, \mathbf{k}^-$ follow from assumption 2.5. □

Proposition 2.2 (Existence of Solutions). *Let $L^1_R := \{c \in L^1(0,T), \int_0^T c^+ \leq R\}$. Assume $u \in C^1(\mathbb{R}^4)$ is concave, verifies the assumption 2.5 and that there exists $C \in \mathbb{R}^*_+$ such that*

$$\forall i \in [\![1,4]\!], \left|\frac{\partial u}{\partial x_i}(x_1, ...x_4)\right| \leq C\left(1 + \sum_{j=1}^{4} |x_j|^{\alpha_j}\right).$$

Assume $\mathbf{p} \in L^\infty(0,T)$ and F and g are sub-linear (assumptions 2.3 and 2.4) and such that

- *$\forall k \in \mathbb{R}, g(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,*
- *$\forall k \in \mathbb{R}, F(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,*
- *F is increasing in the first variable and $(k,t) \mapsto F(k, \mathbf{p}_t)$ is locally uniformly integrable in the second variable.*

Let $P := \|\mathbf{p}\|_{L^\infty(0,T)}$ and $R \in \mathbb{R}$ be such that $R \geq \|g(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}P + \|F(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}$. Let

$$\mathcal{S}(\mathbf{c}, \mathbf{i}, a_0, k_0) := \begin{cases} \int_0^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s & if\ \tau = T \\ -\infty & if\ \mathbf{k} \underset{\tau}{\to} -\infty\ for\ \tau < T \end{cases}$$

where $\mathbf{a}$ and $\mathbf{k}$ and τ are given by the Cauchy problems of assumptions 2.1 and 2.2. Then, if $(a_0, k_0) \in \mathbb{R}^2$ is given, there exists a maximizer to the problem

$$\sup_{(\mathbf{c},\mathbf{i}) \in L^1_R \times L^1(0,T)} \mathcal{S}(\mathbf{c}, \mathbf{i}, a_0, k_0). \tag{2.2}$$

Proof

Let $(\mathbf{c}^n, \mathbf{i}^n)_{n\in\mathbb{N}} \in \mathcal{A}^\mathbb{N}$ be a maximizing sequence and write for all $t \in [0,T]$, $x_1^n(t) := \mathbf{c}_t^n, x_2^n(t) := \mathbf{i}_t^n, x_3^n(t) := \mathbf{a}_t^n, x_4^n(t) := \mathbf{k}_t^n$ and $\mathbf{x}^n := (x_1^n, x_2^n, x_3^n, x_4^n)$. By Proposition 2.1.3, there exists $(\bar{\mathbf{c}}, \bar{\mathbf{i}}) \in L^1(0,T)^2$ such that

$$U_0 := \mathcal{S}(\bar{\mathbf{c}}, \bar{\mathbf{i}}, a_0, k_0) > -\infty.$$

Let $t \in [0,T]$. As $|\bar{\mathbf{c}}_t| = |g(k_0, \mathbf{p}_t)\mathbf{p}_t + F(k_0, \mathbf{p}_t)| \leq R$, we obtain that $\bar{\mathbf{c}} \in L^1_R$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \mathcal{S}(\mathbf{c}^n, \mathbf{i}^n, a_0, k_0) \geq -|U_0|.$$

Let $n \in \mathbb{N}\backslash[\![0, n_0]\!]$. As $\mathcal{S}(\mathbf{c}^n, \mathbf{i}^n, a_0, k_0) > -\infty$, then for all $t \in [0,T[, \mathbf{k}^n$ is bounded by below on $[0,t[$. As g is sub-linear (assumption 2.3) and $\mathbf{p} \in L^\infty(0,T)$, for a.e. $t \in [0,T]$, $\forall k \in \mathbb{R}, g(k, \mathbf{p}_t) \leq \|g_1\|_{\mathcal{L}^\infty(-P,P)} + k\|g_2|_{\mathcal{L}^\infty(-P,P)}$ and by Lemma 2.1.1, $\mathbf{k}^n$ is bounded by above on $[0,\tau[$. Thus, by Theorem B.3, $\mathbf{k}^n$ is well defined on $[0,T[$ and by Lemma 2.1.2, $\mathbf{a}^n \in C^0[0,T[$ and $(\mathbf{a}^n)^+$ is bounded.
Thus, we have

$$((\mathbf{a}^n)^+, (\mathbf{k}^n)^+) \in L^{\gamma_3}(0,T) \times L^{\gamma_4}(0,T).$$

So, we can apply Proposition 2.1, what gives that for all $j \in [\![1,4]\!]$, $(x_j^n)_{n\geq n_0}$ is a bounded sequence of $L^{\alpha_j}(0,T)$. Thus, we can extract weakly convergent subsequences $x_j^{n_m} \overset{L^{\alpha_j}(0,T)}{\longrightarrow} x_j^\infty$. Denote by p' the conjugate exponent of p : $p' := \dfrac{p}{p-1}$. Then for $j \in [\![1,4]\!]$, as $\alpha'_j > 1$,

$$\begin{aligned} \int_0^T \left|\frac{\partial u}{\partial x_j}(\mathbf{x}^n(s))\right|^{\alpha'_j} \mathrm{d}s &\leq \left(\int_0^T \left|\frac{\partial u}{\partial x_j}(\mathbf{x}^n(s))\right| \mathrm{d}s\right)^{\alpha'_j} \\ &\leq \left(CT + \sum_{l=1}^{4} \|x_l^n\|^{\alpha_l}_{L^{\alpha_l}(0,T)}\right)^{\alpha'_j}. \end{aligned}$$

Thus $\left(t \mapsto \frac{\partial u}{\partial x_j}(\mathbf{x}^n(t))\right) \in L^{\alpha'_j}(0,T)$.

As u is concave, we have,

$$\int_0^T u(\mathbf{x}^{n_m}(s))\mathrm{d}s \le \int_0^T u(\mathbf{x}^\infty(s))\mathrm{d}s + \sum_{j=1}^4 \int_0^T \frac{\partial u}{\partial x_j}(\mathbf{x}^\infty(s))(x_j^{n_m} - x_j^\infty)\mathrm{d}s.$$

By weak convergence, $\sum_{j=1}^4 \int_0^T \partial_{x_j} u(\mathbf{x}^\infty(s))(x_j^{n_m} - x_j^\infty)\mathrm{d}s \underset{m\to\infty}{\to} 0$, and as $\mathbf{x}^{n_m}$ is a maximizing sequence of $\mathcal{S}$, $\mathbf{x}^\infty$ is a maximizer of the problem (2.2).

In order to prove that it is the solution of the problem, we have to show that $(\mathbf{c}^\infty, \mathbf{i}^\infty) \in L^1_R \times L^1(0,T)$. It is clear for $\mathbf{i}^\infty$ as $\mathbf{i}^\infty \in L^{\alpha_2}(0,T) \subset L^1(0,T)$. In order to show that $\mathbf{c}^\infty \in L^1_R$, we will show that $L^1_R \subset L^{\alpha_1}(0,T)$ is a closed and convex subspace, and that consequently it is weakly closed.

Let $(v_1, v_2) \in (L^1_R)^2$. By convexity of $(\cdot)^+$, for any $\lambda \in [0,1]$

$$\int_0^T (\lambda v_1 + (1-\lambda)v_2)^+ \le \lambda \int_0^T v_1^+ + (1-\lambda)\int_0^T v_2^+ \le R,$$

so L^1_R is a convex space.

Let $(v_n)_{n\in\mathbb{N}} \in (L^1_R)^{\mathbb{N}}$ such that $v_n \underset{n\to+\infty}{\overset{L^{\alpha_1}}{\longrightarrow}} v_\infty$. For $n \in \mathbb{N}$

$$\int_0^T u_\infty^+ = \int_0^T (u_n + u_\infty - u_n)^+ \le \int_0^T u_n^+ + \int_0^T |u_\infty - u_n|,$$

and using Holder's inequality, we get

$$\int_0^T u_\infty^+ \le R + T^{1/\alpha'_1}\|u_\infty - u_n\|_{L_1^\alpha} \underset{n\to+\infty}{\longrightarrow} R.$$

Thus, L^1_R is convex and closed in $L^{\alpha_1}(0,T)$, so it is closed for in the weak convergence sense. Consequently, as $\mathbf{c}^{n_m} \underset{m\to+\infty}{\overset{L^{\alpha_1}(0,T)}{\rightharpoonup}} \mathbf{c}^\infty$ and $\mathbf{c}^{n_m} \in L^{\alpha_1}(0,T)$, then $\mathbf{c}^\infty \in L^1_R$. □

Corollary 2.1. *With the same hypothesis, we can show the existence of a maximizer to the problem*

$$\sup_{(\mathbf{c},\mathbf{i})\in\mathcal{A}\times L^1(0,T)} \mathcal{S}(\mathbf{c}, \mathbf{i}, a_0, k_0). \tag{2.3}$$

where $\mathcal{A} = \{c \in L^1(0,T), c \le TR\}$.

2.1.2 Existence for c and i Locally Integrable on $[0, T[$

Proposition 2.3 (Bounds on the Trajectories)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be sub-linear (assumptions 2.3 and 2.4), $(\mathbf{c}, \mathbf{i}) \in L^1_{loc}([0,T[)^2$ and $(\mathbf{a}, \mathbf{k}) \in C^0([0,T[)$ absolutely continuous be defined by assumptions 2.1 and 2.2 and be such that $\mathbf{c}^+ \in L^1(0,T)$ and there exists $U_0 \in \mathbb{R}_+$ such that $\forall t \in [0,T[, \int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s > -U_0$. Let u verify assumption 2.5 and $\mathbf{p} \in L^\infty(0,T)$.*

Then, we have

$$(\mathbf{c}^+, \mathbf{a}^+, \mathbf{k}^+) \in \mathcal{L}^\infty(0,T)^3,$$

$$\mathbf{c}^- \in L^{\alpha_1}(0,T),\ \mathbf{i} \in L^{\alpha_2}(0,T),\ \mathbf{a}^- \in L^{\alpha_3}(0,T),\ \mathbf{k}^- \in L^{\alpha_4}(0,T).$$

Proof

Define $R := \|c^+\|_{L^1(0,T)}$, $A : t \mapsto \int_0^t (\mathbf{a}^+)^{\gamma_3}$, $K : t \mapsto \int_0^t (\mathbf{k}^+)^{\gamma_3}$, $I : t \mapsto \int_0^t |\mathbf{i}|$ and $U_R := 2T + R + U_0 + C_u^{-1}$.

(i) There exists $(C_{2,R}, C_2) \in \mathbb{R}_+^2, \mathbf{k}_t \leq C_{2,R} + C_2 A_t$.

As $\mathbf{p} \in L^\infty(0,T)$, by sub-linearity of g (assumption 2.3), there exists $(G_0, G_1) \in \mathbb{R}^2$ such that for a.e. $t \in [0,T[$, $g(\mathbf{k}_t, \mathbf{p}_t) \leq G_0 \mathbf{k}_t + G_1$, so such that $\dot{\mathbf{k}}_t = \mathbf{i}_t + g(\mathbf{k}_t, \mathbf{p}_t) \leq \mathbf{i}_t + G_0 \mathbf{k}_t + G_1$

Let $t \in [0, T[$. As in the proof of Proposition 2.1, we have

$$I_t \leq U_R + A_t + \int_0^t (C_0 + \widetilde{C_0} I_s)^{\gamma_3} \mathrm{d}s.$$

Thus, by using Lemma C.1.2 with $\alpha = 1$, we have

$$I_t \leq U_R + A_t + (1 + C_0)t + \widetilde{C_0} \int_0^t I_s \mathrm{d}s.$$

Let $\bar{C}_0 := \widetilde{C_0} e^{\widetilde{C_0} T}$. Then by the Gronwall's integral inequality (Proposition A.3) and using the fact that $\int_0^t A < t A_t$ because A is non-negative and increasing, we have

$$\begin{aligned} I_t &\leq U_R + A_t + (1 + C_0)t + \bar{C}_0 \int_0^t (U_R + A_s + (1 + C_0)s) \mathrm{d}s \\ &\leq A_t(1 + \bar{C}_0 t) + U_R + (1 + C_0 + U_R \bar{C}_0)t + \bar{C}_0(1 + C_0)\frac{t^2}{2} \\ &\leq C_1 A_t + C_{1,R} \end{aligned}$$

where $C_1 = 1 + \bar{C}_0 T$ and $C_{1,R} = U_R + (1 + C_0 + U_R \bar{C}_0)T + \bar{C}_0(1 + C_0)\dfrac{T^2}{2}$.

So we obtain $\forall t \in [0, T[, \mathbf{k}_t \leq C_0 + \widetilde{C_0} C_{1,R} + \widetilde{C_0} C_1 A_t$

(ii) There exists $(C_{3,R}, C_3) \in \mathbb{R}_+^2$, $\int_0^t \mathbf{k}^- \leq C_{3,R} + C_3 A_t$ and $\int_0^t \mathbf{c}^- \leq C_{3,R} + C_3 A_t$.

As u verifies assumption 2.5, we have

$$\int_0^t \mathbf{k}^- \leq U_R + A_t + \int_0^t (C_{2,R} + C_2 A_s)^{\gamma_3} \mathrm{d}s.$$

Thus by using Lemma C.1.2 for $\alpha = 1$, we have

$$\int_0^t (\mathbf{k}^-) \leq t + U_R + A_t + (1 + C_{2,R})t + C_2 t A_t$$

Thus we have the result for $C_3 := 1 + C_2 T$ and $C_{3,R} := (2 + C_{2,R})T + U_R$.

The same demonstration holds for $\mathbf{c}^-$, by replacing α_4 by α_1.

(iii) There exists $(C_{4,R}, C_4) \in \mathbb{R}_+^2, \mathbf{a}_t^+ \leq C_{4,R} + C_4 A_T$.

As for g, the sub-linearity of F (assumption 2.4), gives two constants $(F_0, F_1) \in \mathbb{R}^2$ such that for a.e. $t \in [0, T[$, $F(\mathbf{k}_t, \mathbf{p}_t) \leq F_0 \mathbf{k}_t + F_1$.

$$\begin{aligned} \mathbf{a}_t &= a_0 - \int_0^t (\mathbf{c} + \mathbf{p}\mathbf{i}) + \int_0^t F(\mathbf{k}_s, \mathbf{p}_s) \mathrm{d}s \\ &\leq |a_0| + \int_0^t \mathbf{c}^- + R + \|\mathbf{p}\|_{L^\infty} \int_0^t |\mathbf{i}| + F_1 t + F_0 \int_0^t \mathbf{k} \\ &\leq |a_0| + C_{3,R} + C_3 A_t + R + \|\mathbf{p}\|_{L^\infty}(1 + C_1 A_t + C_{1,R}) + t(F_1 + F_0(C_{2,R} + C_2 A_t)) \\ &\leq C_{4,R} + C_4 A_t \end{aligned}$$

where $C_4 = C_3 + \|\mathbf{p}\|_{L^\infty} C_1 + F_0 C_2 T$ and $C_{4,R} = |a_0| + C_{3,R} + R + \|\mathbf{p}\|_{L^\infty}(1 + C_{1,R}) + (F_1 + F_0 C_{2,R})T$. And as $C_{4,R}, C_4$ and A_t are positive, $\mathbf{a}_t^+ \leq C_{4,R} + C_4 A_t$.

(iv) $\mathbf{a}^+$ and $\mathbf{k}^+$ are bounded

By using Lemma C.1.2 for $\alpha = 1$, we have

$$A_t = \int_0^t (\mathbf{a}^+)^{\gamma_3} \leq t(1 + C_{4,R}) + C_4 \int_0^t A$$

so by using again Gronwall's integral inequality (Proposition A.3), we obtain

$$A_t \leq (1 + C_{4,R})T + C_4 e^{C_4 T}(1 + C_{4,R})\frac{T^2}{2}$$

Thus A_t is bounded hence (i) and (iii) show that $\mathbf{a}^+$ and $\mathbf{k}^+$ are bounded.

The bounds on $\mathbf{c}^-, |\mathbf{i}|, \mathbf{a}^-, \mathbf{k}^-$ follow from assumption 2.5. □

Theorem 2.1 (Existence of Solutions). *Let $L^1_{R,loc} := \{c \in L^1_{loc}([0,T[), \int_0^T c^+ \leq R\}$. Assume $u \in C^1(\mathbb{R}^4)$ is concave, verifies the assumption 2.5 and that there exists $C \in \mathbb{R}^*_+$ such that*

$$\forall i \in [\![1,4]\!], \left|\frac{\partial u}{\partial x_i}(x_1, ...x_4)\right| \leq C\left(1 + \sum_{j=1}^4 |x_j|^{\alpha_j}\right).$$

Assume $\mathbf{p} \in L^\infty(0,T)$, F and g are sub-linear (assumptions 2.3 and 2.4) and such that

- $\forall k \in \mathbb{R}, g(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,
- $\forall k \in \mathbb{R}, F(k,\cdot) \in \mathcal{L}^\infty_{loc}(\mathbb{R})$,
- *$(k,t) \mapsto F(k,\mathbf{p}_t)$ is locally uniformly integrable in the second variable.*

Let $P := \|\mathbf{p}\|_{L^\infty(0,T)}$ and $R \in \mathbb{R}$ be such that $R \geq \|g(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}P + \|F(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}$, $\mathcal{S}$ be defined as in Proposition 2.2 and $(a_0,k_0) \in \mathbb{R}^2$ be given. Then there exists a maximizer to the problem

$$\sup_{(\mathbf{c},\mathbf{i}) \in L^1_{R,loc} \times L^1_{loc}([0,T[)} \mathcal{S}(\mathbf{c},\mathbf{i},a_0,k_0). \tag{2.4}$$

Proof

Let $(\mathbf{c}^n, \mathbf{i}^n)_{n\in\mathbb{N}} \in (L^1_{R,loc} \times L^1_{loc}([0,T[))^{\mathbb{N}}$ be a maximizing sequence. As in the proof of Proposition 2.2, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \mathcal{S}(\mathbf{c}^n, \mathbf{i}^n, a_0, k_0) \geq -|U_0|$$

Let $n \in \mathbb{N}\backslash[\![0,n_0]\!]$. As $\mathcal{S}(\mathbf{c}^n, \mathbf{i}^n, a_0, k_0) > -\infty$, then for all $t \in [0,T[, \mathbf{k}^n$ is bounded by below on $[0,t[$. As g is sub-linear (assumption 2.3) and $\mathbf{p}$ is bounded, for a.e. $t \in [0,T]$, $\forall k \in \mathbb{R}, g(k,\mathbf{p}_t) \leq \|g_1\|_{\mathcal{L}^\infty(-P,P)} + k\|g_2\|_{\mathcal{L}^\infty(-P,P)}$ and by Lemma 2.1.1, $\mathbf{k}^n$ is bounded by above on $[0,\tau[$. Thus, by Theorem B.3, $\mathbf{k}^n$ is well defined on $[0,T[$ and by Lemma 2.1.2, $\mathbf{a}^n \in C^0([0,T[)$. Let

$$S^n : t \mapsto \int_0^t u(\mathbf{c}^n_s, \mathbf{i}^n_s, \mathbf{a}^n_s, \mathbf{k}^n_s)\mathrm{d}s.$$

As $S \in C^0([0,T[)$ and $S(T) \in \mathbb{R}\cup\{+\infty\}$ is such that $S(T) \geq -|U_0|$, there exists $U^n_0 \in \mathbb{R}_+$ such that $\forall t \in [0,T], S^n(t) > -U^n_0$. Then by proposition 2.3, we obtain that $(\mathbf{c}^n, \mathbf{i}^n) \in L^1(0,T)$ and $(\mathbf{a}^n)^+ \in \mathcal{L}^\infty(0,T)$ and we can use the first existence proposition (Proposition 2.2) to show the result (and we do not need F increasing in the first variable because we already have $(\mathbf{a}^n)^+$ bounded). □

Corollary 2.2 (Existence of Solutions - F and g Concave). *Let $L^1_{R,loc} := \{c \in L^1_{loc}(0,T), \int_0^T c^+ \leq R\}$. Assume $u \in C^1(\mathbb{R}^4)$ is concave, verifies the assumption 2.5 and that there exists $C \in \mathbb{R}^*_+$ such that*

$$\forall i \in [\![1,4]\!], \left|\frac{\partial u}{\partial x_i}(x_1,...x_4)\right| \leq C\left(1+\sum_{j=1}^4 |x_j|^{\alpha_j}\right).$$

Assume F and g are concave and $\mathbf{p} \in L^\infty(0,T)$.
Let $P := \|\mathbf{p}\|_{L^\infty(0,T)}$ and $R \in \mathbb{R}$ be such that $R \geq \|g(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}P + \|F(k_0,\cdot)\|_{\mathcal{L}^\infty(-P,P)}$, $\mathcal{S}$ be defined as in Proposition 2.2 and $(a_0,k_0) \in \mathbb{R}^2$ be given. Then there exists a maximizer to the problem

$$\sup_{(\mathbf{c},\mathbf{i})\in L^1_{R,loc}\times L^1_{loc}([0,T[)} \mathcal{S}(\mathbf{c},\mathbf{i},a_0,k_0). \tag{2.5}$$

Proof
As F and g are concave, they are sub-linear (assumptions 2.4 and 2.3) and, by the proposition C.2, $(F,g) \in \mathcal{L}^\infty_{loc}(\mathbb{R}^2)^2$. As $\mathbf{p} \in L^\infty(0,T)$, for $K \subset \mathbb{R}$ compact, for a.e. $t \in [0,T]$, $\sup_{k\in K}|F(k,\mathbf{p}_t)| \leq M$. Consequently, $(k,t) \mapsto F(k,\mathbf{p}_t)$ is locally uniformly integrable in the second variable, and using the previous theorem, we obtain the result. □

2.2 Conditions to have $\mathbf{k}^-$ bounded

In the two previous sections, to have a well-defined problem, we defined the functional to be maximized as $-\infty$ if $\mathbf{k}^- \to -\infty$ for $t < T$. Then, we obtain the existence of a maximizer, but we can still have $\mathbf{k}^-$ blowing up at $t = T$. In this section, we look at conditions that ensure that $\mathbf{k}$ does not blow up for $t \leq T$ and conditions for which $\mathbf{k}$ does not blow up at $t = T$. It can be useful to create models if we know that the capital should be bounded and also for numerical purposes.

Lemma 2.2.1 (Local Definiteness of $\mathbf{k}$). *Let $\mathbf{i} \in L^1_{loc}([0,T[)$, $\mathbf{p} \in L^\infty(0,T)$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be locally uniformly Lipschitz in the first variable and such that $(k,t) \mapsto g(k,\mathbf{p}_t)$ is locally uniformly integrable in the second variable.*

Then there exists $\tau \in]0,T]$ and $\mathbf{k} \in C^0[0,\tau[$ absolutely continuous satisfying assumption 2.1.

Proof
The result is an application of Cauchy-Lipschitz theorem for locally integrable functions (Theorem B.2). □

Lemma 2.2.2. *Let $\mathbf{k}$ and τ be defined by assumption 2.1 with $\mathbf{i} \in L^1(0,T)$.*

Assume $\mathbf{k}^+ \in L^\infty(0,\tau)$ and that there exists $(G_3,G_4) \in \mathbb{R}^2$ such that $\forall (k,t) \in \mathbb{R}_- \times [0,\tau[, g(k,\mathbf{p}_t) \geq G_3 k + G_4$.

Then $\tau = T$ and $\mathbf{k} \in C^0[0,T]$.

Proof
Suppose $\mathbf{k}$ is not defined on $[0,T]$. Then, as $\mathbf{k}$ is bounded by above, by Theorem B.3, $\mathbf{k}_t \underset{t\to\tau}{\to} -\infty$. Thus, there exists $t_1 \in [0,\tau[$ such that $\forall t \in [t_1,\tau[, \mathbf{k}_t \leq 0$.

Let $t \in [t_1,\tau[$. As $\dot{\mathbf{k}}_t \geq \mathbf{i}_t + G_3\mathbf{k}_t + G_4$. By applying Gronwall's Inequality (Proposition A.1), we obtain:

$$\begin{aligned}
\mathbf{k}_t &\geq e^{G_3 t}\left(k_0 + \int_0^t (\mathbf{i}_s + G_4)e^{-G_3 s}\mathrm{d}s\right) \\
&\geq -e^{G_1 t}\left(k_0 + G_2G_1^{-1}(1-e^{-G_1 t}) + \int_0^t |\mathbf{i}_s|e^{-G_1 s}\mathrm{d}s\right) \\
&\geq -e^{G_1\tau}\left(|k_0| + |G_2G_1^{-1}|(1+e^{|G_1|\tau}) + \|\mathbf{i}\|_{L^1(0,T)}e^{|G_1|\tau}\right),
\end{aligned}$$

which is a contradiction □

Lemma 2.2.3. *Let* $\mathbf{k}$ *and* τ *be defined by assumption 2.1 with* $\mathbf{i} \in L^1(0,T)$.

Assume $\mathbf{k}^+ \in L^\infty(0,\tau)$ *and there exists* $\alpha_4 \in [1,+\infty[$ *such that* $\mathbf{k} \in L^{\alpha_4}(0,\tau)$ *and* $(G_3, G_4) \in \mathbb{R}^2$ *such that* $\forall (k,t) \in \mathbb{R}_- \times [0,\tau[, g(k,\mathbf{p}_t) \geq G_3|k|^{\alpha_4} + G_4$.

Then $\tau = T$ *and* $\mathbf{k} \in C^0[0,T]$.

Proof

Suppose $\mathbf{k}$ is not defined on $[0,T]$. As in the previous proof, by Theorem B.3, $\mathbf{k}_t \underset{t\to\tau}{\to} -\infty$ and there exists $t_1 \in [0,\tau[$ such that $\forall t \in [t_1,\tau[, \mathbf{k}_t \leq 0$.

Let $t \in [t_1,\tau[$, then $\dot{\mathbf{k}}_t \geq \mathbf{i}_t + G_3|\mathbf{k}_t|^{\alpha_4} + G_4$. Thus, by integrating in time

$$\begin{aligned} \mathbf{k}_t &\geq \mathbf{k}_{t_1} + \int_{t_1}^t \mathbf{i} + G_3 \int_{t_1}^t |\mathbf{k}|^{\alpha_4} + G_4 \\ &\geq \mathbf{k}_{t_1} - \|\mathbf{i}\|_{L^1(0,T)} - |G_3| \|\mathbf{k}\|_{L^{\alpha_4}}^{\alpha_4} + G_4. \end{aligned}$$

As $\mathbf{k} \in C^0[0,\tau[$, if $t_1 \neq 0$, then $\mathbf{k}_{t_1} = 0 \geq -|k_0|$. If $t_1 = 0$, then $\mathbf{k}_{t_1} \geq -|k_0|$. Thus, we have

$$\mathbf{k}_t \geq -|k_0| - \|\mathbf{i}\|_{L^1(0,T)} - |G_3| \|\mathbf{k}\|_{L^{\alpha_4}}^{\alpha_4} + G_4,$$

which is a contradiction. □

Lemma 2.2.4. *Let* $\mathbf{k}$ *and* τ *be defined by assumption 2.1 with* $\mathbf{i} \in L^1(0,T)$.

Assume $\mathbf{k}^+ \in L^\infty(0,T)$ *and there exists* $(G_3, G_4, \alpha) \in \mathbb{R}_+^2 \times]1,+\infty[$ *such that* $\forall (k,t) \in \mathbb{R}_- \times]0,\tau[, g(k,\mathbf{p}_t) \geq -G_3|k|^\alpha - G_4$.

If $k_0 \neq 0$, *then* $\tau \geq \min\left(T, Ce^{-\frac{\alpha-1}{|k_0|}(\|\mathbf{i}^-\|_{L^1(0,T)} + G_4 T)}\right)$ *where* $C^{-1} = (\alpha-1)G_3|k_0|^{\alpha-1}$.

If $k_0 \geq -\|\mathbf{i}^-\| - G_4 T$, *then* $\tau \geq \min\left(T, \dfrac{e^{-(\alpha-1)}}{(\alpha-1)G_3(\|\mathbf{i}^-\|_{L^1(0,T)} + G_4 T)^{\alpha-1}}\right)$.

Proof

<u>**Case where** $k_0 \neq 0$</u>

Assume $\tau < T$. Let $\widetilde{k}$ be the maximal solution to the Cauchy problem

$$\begin{cases} \dot{\widetilde{k}}_t &= -\mathbf{i}_t^- - g(\widetilde{k}_t, \mathbf{p}_t)^- \\ \widetilde{k}_0 &= -|k_0|, \end{cases}$$

and $\tau_2 := \sup(t \in [0,T], \widetilde{k}$ is defined on $[0,t[)$. By the ODE comparison principle, $\widetilde{k} \leq \mathbf{k}$ and $\widetilde{k} \leq 0$. Thus, we have for all $t \in [0,\tau_2[, g(\widetilde{k}_t,\mathbf{p}_t) \geq -G_3|\widetilde{k}_t|^\alpha - G_4$ and as $G_3 \geq 0$ and $G_4 \geq 0$, we also have $-g(\widetilde{k}_t,\mathbf{p}_t)^- \geq -G_3|\widetilde{k}_t|^\alpha - G_4$. Thus, we obtain

$$\dot{\widetilde{k}}_t \geq -\mathbf{i}_t^- - G_3|\widetilde{k}|^\alpha - G_4.$$

Let $\overline{k} := -G_3^{\frac{1}{\alpha-1}}\widetilde{k}$ and $j : t \mapsto (\mathbf{i}_t^- + G_4)G_3^{\frac{1}{\alpha-1}} \in L^1(0,T)$. Now we have

$$\dot{\overline{k}}_t \leq j_t + \overline{k}_t^\alpha \text{ and } \overline{k} \geq 0.$$

Let $\bar{v}_t$ be the maximal solution to $\dot{\bar{v}}_t = j_t + \bar{v}_t^\alpha$ with $\bar{v}_0 = \overline{k}_0$ and $\tau_3 := \min(\tau_2, \sup(t \in [0,T], \bar{v}$ is defined on $[0,t[))$. By the ODE comparison principle, we obtain for all $t \in [0,\tau_3[$,

$$\bar{v}_t \geq \bar{k}_t \geq 0.$$

$\dot{\bar{v}} \geq 0$ implies that $\bar{v}$ is increasing, thus we have $\dot{\bar{v}}_t \leq \dfrac{j_t}{\bar{v}_0}\bar{v}_t + \bar{v}_t^\alpha$. Let v be the solution to $\dot{v}_t = \dfrac{j_t}{\bar{v}_0}v_t + v_t^\alpha, w := v^{1-\alpha}$ and $C^{-1} := \overline{k}_0^{\alpha-1}(\alpha - 1)$.

$$\begin{aligned}
\dot{v}_t = \frac{j_t}{\bar{v}_0}v_t + v_t^\alpha \quad &\Leftrightarrow \quad \frac{\dot{w}_t}{1-\alpha}w_t^{\frac{\alpha}{1-\alpha}} = \frac{j_t}{\bar{v}_0}w_t^{\frac{1}{1-\alpha}} + w_t^{\frac{\alpha}{1-\alpha}} \\
&\Leftrightarrow \quad \dot{w}_t = -(\alpha-1)\frac{j_t}{\bar{v}_0}w_t + 1 - \alpha \\
&\Leftrightarrow \quad w_t = e^{-\frac{\alpha-1}{\bar{v}_0}\int_0^t j}\left(w_0 - (\alpha-1)\int_0^t e^{\frac{\alpha-1}{\bar{v}_0}\int_0^s j}\mathrm{d}s\right) \\
&\Leftrightarrow \quad v_t = e^{\bar{v}_0^{-1}\int_0^t j}\frac{1}{\left(v_0^{1-\alpha} - (\alpha-1)\int_0^t e^{\frac{\alpha-1}{\bar{v}_0}\int_0^s j}\mathrm{d}s\right)^{\frac{1}{\alpha-1}}} \\
&\Leftrightarrow \quad v_t = e^{\bar{v}_0^{-1}\int_0^t j}\frac{v_0}{\left(1 - C^{-1}\int_0^t e^{\frac{\alpha-1}{\bar{v}_0}\int_0^s j}\mathrm{d}s\right)^{\frac{1}{\alpha-1}}}.
\end{aligned}$$

Consequently, v is defined on $[0,\tau_4[$ where τ_4 is such that $\int_0^{\tau_4} e^{\frac{\alpha-1}{\bar{v}_0}\int_0^s j}\mathrm{d}s = C$. As $j \in L^1(0,T)$, we have $\tau_4 \geq Ce^{-\frac{\alpha-1}{\overline{k}_0}\|j\|_{L^1}}$, and by using the ODE comparison principle and the Theorem B.3, we obtain that $0 \leq \overline{k} \leq \bar{v} \leq v$ and $\tau_2 \geq \tau_3 \geq \tau_4$. As $-\overline{k} = G_3^{\frac{1}{\alpha-1}}\widetilde{k} \leq G_3^{\frac{1}{\alpha-1}}\mathbf{k} \leq M$, we can use one more time the Theorem B.3 to obtain that $\tau \geq \tau_2 \geq Ce^{-\frac{\alpha-1}{\overline{k}_0}\|j\|_{L^1(0,T)}} \geq Ce^{-\frac{\alpha-1}{|k_0|}(\|\mathbf{i}^-\|_{L^1(0,T)}+G_4T)}$.

Case where $k_0 \geq -\|\mathbf{i}^-\| - G_4T$
Assume $\tau < T$. Then, as $\mathbf{k} \leq M$, by Theorem B.3, $\mathbf{k} \underset{\tau}{\rightarrow} -\infty$ and as $\mathbf{k} \in C^0([0,\tau[)$, there exists $t_1 \in]0,\tau[$ such that $\mathbf{k}_{t_1} = -\|\mathbf{i}^-\| - G_4T$. By the first part of the demonstration, we obtain that

$$\tau \geq \frac{1}{(\alpha-1)G_3|\mathbf{k}_{t_1}|^{\alpha-1}}e^{-\frac{\alpha-1}{|\mathbf{k}_{t_1}|}(\|\mathbf{i}^-\|_{L^1(0,T)}+G_4T)} \geq \frac{1}{(\alpha-1)G_3(\|\mathbf{i}^-\|_{L^1(0,T)}+G_4T)^{\alpha-1}}e^{-(\alpha-1)}$$

□

Proposition 2.4. *Let $(\mathbf{c},\mathbf{i}) \in L^1(0,T)^2$, $\mathbf{p} \in L^\infty(0,T)$, g be locally uniformly Lipschitz in the first variable, sub-linear (assumption 2.3) and satisfy the bound from assumption 2.6 for $\alpha \in]1,+\infty[$ and $(k,t) \mapsto g(k,\mathbf{p}_t)$ and $(k,t) \mapsto F(k,\mathbf{p}_t)$ be locally uniformly integrable in the second variable.*

Then, there exists $\mathbf{a} \in C^0[0,\tau[$ and $\mathbf{k} \in C^0[0,\tau[$ absolutely continuous satisfying the differential equations from assumptions 2.2 and 2.1 with $\tau \geq \min\left(T, Ce^{-\frac{\alpha-1}{|k_0|}(\|\mathbf{i}^-\|_{L^1(0,T)}+G_4T)}\right)$ if $k_0 \neq 0$ (where $C^{-1} = (\alpha-1)G_3|k_0|^{\alpha-1}$), and with $\tau \geq \min\left(T, \dfrac{e^{-(\alpha-1)}}{(\alpha-1)G_3(\|\mathbf{i}^-\|_{L^1(0,T)}+G_4T)^{\alpha-1}}\right)$ if $k_0 \geq -\|\mathbf{i}^-\|_{L^1} - G_4T$.

Moreover, if one of the following hypothesis is fulfilled,

- *g satisfy assumption 2.6 for $\alpha \in [0,1]$,*
- *$\mathbf{k} \in L^\alpha(0,\tau)$,*

- $\|\mathbf{i}^-\|_{L^1(0,T)} < \dfrac{|k_0|}{\alpha-1}\ln\left(\dfrac{C}{T}\right) - G_4 T$,

- $-k_0 - G_4 T \leq \|\mathbf{i}^-\|_{L^1(0,T)} < \dfrac{e^{-1}}{(\alpha-1)G_3 T} - G_4 T$,

then $\tau = T$ *and* $(\mathbf{a}, \mathbf{k}) \in C^0[0,T]^2$.

Notice that if $\mathbf{k}$ *is a solution to the maximizing problem of Theorem 2.1, and* $\alpha_4 \geq \alpha$*, then the second hypothesis is fulfilled and we obtain that this optimal* $\mathbf{k}$ *is bounded.*

Proof

Let $P := [-\|p\|_\infty, \|p\|_\infty]$. By lemmas 2.1.2 and 2.2.1 there exists $\mathbf{a}, \mathbf{k}, \tau$ absolutely continuous satisfying assumptions 2.1 and 2.2.

By assumption 2.3, there exists $(g_1, g_2) \in \mathcal{L}^\infty_{loc}(\mathbb{R})^2$ such that $\forall (k,p) \in \mathbb{R}^2, g(k,p) \leq g_1(p)k + g_2(p)$. Consequently, for $(k,t) \in \mathbb{R} \times [0,T]$, we have $g(k, \mathbf{p}_t) \leq \|g_1\|_{\mathcal{L}^\infty(P)} k + \|g_2\|_{\mathcal{L}^\infty(P)}$, and by applying Lemma 2.1.1, we obtain that $\mathbf{k}$ is bounded by above on $[0, \tau[$.

By assumption 2.6, there exists $(g_3, g_4, \alpha) \in \mathcal{L}^\infty_{loc}(\mathbb{R})^2 \times]1, +\infty[$ such that $\forall (k,p) \in \mathbb{R}_- \times \mathbb{R}, g(k,p) \geq g_3(p)|k|^\alpha + g_4(p)$. Consequently, for $(k,t) \in \mathbb{R}_- \times [0,T]$, we have $g(k, \mathbf{p}_t) \geq -G_3|k|^\alpha - G_4$ where $G_3 = \|g_3\|_{\mathcal{L}^\infty(P)}$ and $G_4 = \|g_4\|_{\mathcal{L}^\infty(P)}$. Thus, as $\mathbf{k}$ is bounded by above, by applying Lemma 2.2.4, we obtain the bounds for τ.

If g satisfy assumption 2.6 for $\alpha \in [0,1]$, then by Lemma C.1.2, for $(k,t) \in \mathbb{R}_- \times [0,T]$, $g(k, \mathbf{p}_t) \geq -G_3|k|^\alpha - G_4 \geq -G_3|k| - G_4 - 1$. Consequently, Lemma 2.2.2 shows that $\tau = T$ and $\mathbf{k} \in C^0[0,T]$ and Lemma 2.1.2 shows that $\mathbf{a} \in C^0[0,T]$.

If $\mathbf{k} \in L^\alpha(0,\tau)$, then Lemma 2.2.3 shows that $\tau = T$ and $\mathbf{k} \in C^0[0,T]$ and Lemma 2.1.2 shows that $\mathbf{a} \in C^0[0,T]$. $\square$

With the previous proposition, we can build a new existence proposition in which there is no risk of small-time-blowing solutions. In the following proposition, it is the hypothesis on g, i.e. g verifies assumption 2.6 for $\alpha \in [0,1]$, that allows us to be sure that $\mathbf{k}$ will be defined on $[0,T]$ and bounded. An example of g verifying these assumptions is g decreasing. Another example could be $g > 0$ for $k < 0$.

Corollary 2.3 (Existence of Solutions - F and g Concave and g Greater than a Linear Function for a Negative Capital)**.** *Let* $L^1_{R,loc} := \{c \in L^1_{loc}(0,T), \int_0^T c^+ \leq R\}$.
Assume $u \in C^1(\mathbb{R}^4)$ *is concave, verifies the assumption 2.5 and that there exists* $C \in \mathbb{R}_+^*$ *such that*

$$\forall i \in [\![1,4]\!], \left|\frac{\partial u}{\partial x_i}(x_1, ...x_4)\right| \leq C\left(1 + \sum_{j=1}^{4} |x_j|^{\alpha_j}\right).$$

Assume F *and* g *are concave,* g *verifies assumption 2.6 for* $\alpha \in [0,1]$ *and* $\mathbf{p}$ *is bounded.*
Let $P := \|\mathbf{p}\|_{L^\infty(0,T)}$ *and* $R \in \mathbb{R}$ *be such that* $R \geq \|g(k_0, \cdot)\|_{L^\infty(-P,P)} P + \|F(k_0, \cdot)\|_{L^\infty(-P,P)}$, $(a_0, k_0) \in \mathbb{R}^2$ *be given. Then there exists a maximizer* $(\mathbf{c}^*, \mathbf{i}^*, \mathbf{a}^*, \mathbf{k}^*)$ *to the problem*

$$\sup_{(\mathbf{c},\mathbf{i}) \in L^1_{R,loc} \times L^1_{loc}([0,T[)} \int_0^T u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s) \mathrm{d}s, \tag{2.6}$$

where $\mathbf{k}$ *and* $\mathbf{a}$ *are defined by assumptions 2.1 and 2.2.*

Moreover, $(\mathbf{c}^*, \mathbf{i}^*, \mathbf{a}^*, \mathbf{k}^*) \in L^{\alpha_1}(0,T) \times L^{\alpha_2}(C,T) \times C^0([0,T])^2$.

2.3 Examples to Justify Some Hypothesis

2.3.1 c not Bounded

In this section, we show examples of settings that would make the utility blow up in case some assumptions of the previous theorem are not fulfilled. This first example justifies the fact that we need to have a bound on the integral of $\mathbf{c}^+$ if we want the theorem to be valid for all γ_1 and α_3. An interesting fact is that it seems that under some conditions on γ_1 and α_3 the existence could still be proved without assuming any uniform bound on $\mathbf{c}^+$.

Proposition 2.5. *Assume there exists $k_0 \in \mathbb{R}$ such that $g(k_0, \mathbf{p}_t) = 0$ and $F(k_0, \mathbf{p}_t) = 0$ (for example $k_0 = 0$ and $g(0, \cdot) = F(0, \cdot) = 0$) and that $u = u_1(c, a) + u_2(i, k)$ where $u_1(c, a) = (c^+)^{\gamma_1} + (a^+)^{\gamma_3} - (c^-)^{\alpha_1} - (a^-)^{\alpha_3}$.*
Assume $\mathbf{a}$ and $\mathbf{k}$ verify the Cauchy Problems from assumptions 2.2 and 2.1 with $\mathbf{i} = 0$ and $a_0 < 0$, then $\mathbf{k} = k_0$ and $\dot{\mathbf{a}} = -\mathbf{c}$.
If $\dfrac{1}{\gamma_1} - \dfrac{1}{\alpha_3} < 1$, then

$$\exists \mathbf{c} \in L^1_{loc}[0, T[, \int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s \underset{t\to T}{\to} +\infty.$$

Proof

Let $\mathbf{c}_t := \dfrac{\alpha - 1}{(T-t)^\alpha}$ with $\alpha > 1$, then

$$u_1(\mathbf{c}_t, \mathbf{a}_t) = \frac{\alpha-1}{(T-t)^{\alpha\gamma_1}} - \left| a_0 - \int_0^t \frac{\alpha-1}{(T-s)^\alpha}\mathrm{d}s \right|^{\alpha_3} = \frac{\alpha-1}{(T-t)^{\alpha\gamma_1}} - \left(|a_0| + \int_0^t \frac{\alpha-1}{(T-s)^\alpha}\mathrm{d}s \right)^{\alpha_3}.$$

Hence, by convexity of $x \mapsto x^\alpha$, we have

$$u_1(\mathbf{c}_t, \mathbf{a}_t) = \frac{\alpha-1}{(T-t)^{\alpha\gamma_1}} - \left(|a_0| + \frac{1}{(T-t)^{\alpha-1}} - \frac{1}{T^{\alpha-1}} \right)^{\alpha_3} \geq \frac{\alpha-1}{(T-t)^{\alpha\gamma_1}} - \frac{2^{\alpha-1}}{(T-t)^{(\alpha-1)\alpha_3}} + \underset{t\to T}{O}(1).$$

As $\dfrac{1}{\gamma_1} - \dfrac{1}{\alpha_3} < 1$, we have $\dfrac{1}{\gamma_1} < \dfrac{\alpha_3}{\alpha_3 - \gamma_1}$. Thus, we can find an α such that

$$\frac{1}{\gamma_1} \leq \alpha < \frac{\alpha_3}{\alpha_3 - \gamma_1}$$

and, for such an α, we have

$$\alpha\gamma_1 \geq 1 \ \wedge \ (\alpha - 1)\alpha_3 < \alpha\gamma_1$$

consequently

$$\int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s = \int_0^t u_1(\mathbf{c}_s, \mathbf{a}_s)\mathrm{d}s + u_2(0, k_0)t \underset{t\to T}{\to} +\infty$$

□

2.3.2 p not Bounded

Assume $\mathbf{a}$ and $\mathbf{k}$ verify the Cauchy problems from assumptions 2.2 and 2.1 with $g(k_0, \cdot) \in \mathcal{L}^\infty(\mathbb{R})$, $\Theta(k_0) \geq 0$ and $\Xi(k_0) > -\|g^-(k_0, \cdot)\|_{\mathcal{L}^\infty(\mathbb{R})}$ where $F(k_0, p) = \Theta(k_0) + \Xi(k_0)p$, meaning that we can find a capital amount such that the gain in goods is greater than the depreciation in capital. Let write $G := \Xi(k_0) + \|g^-(k_0, \cdot)\|_{\mathcal{L}^\infty(\mathbb{R})} > 0$
Let $\mathbf{c} := 0$ and $\mathbf{i}_t := -g(k_0, \mathbf{p}_t)$ then $\mathbf{k} = k_0$ and $\dot{\mathbf{a}}_t \geq \mathbf{p}_t(g(k_0, \mathbf{p}_t) + \Xi(k_0)) \geq \mathbf{p}_t G > 0$.
Let take $u(i, a, k) = (a^+)^{\gamma_3} + (k^+)^{\gamma_4} - |i|^{\alpha_2} - (a^-)^{\alpha_3} - (k^-)^{\alpha_4}$, then

$$\int_0^t u(i_s, a_s, k_s)\mathrm{d}s > \int_0^t (a_0 + G\mathbf{p}_s)^{\gamma_3}\mathrm{d}s + t(k_0)^{\gamma_4} - \|g(k_0, \cdot)\|^{\alpha_2}_{\mathcal{L}^\infty(\mathbb{R})}.$$

Thus

$$\int_0^t \mathbf{p}^{\gamma_3} \to +\infty \ \Rightarrow \ \int_0^t u(\mathbf{c}_s, \mathbf{i}_s, \mathbf{a}_s, \mathbf{k}_s)\mathrm{d}s \to +\infty.$$

Chapter 3

Computations in the N-Player Case

In this chapter, we assume that there is a fixed number of players $N \in \mathbb{N}^*$ and we denote by $\mathbf{a}_t^n, \mathbf{k}_t^n$ the goods and the capital of the player n at time t. In the previous chapter, we showed that, given the price as a function of the time, there exist optimal trajectories. In the finite player case, knowing these trajectories, we can find an explicit solution to the transport equation 1.4.

But the solution to our mean field game problem also require the equilibrium equation for the price (equation 1.5) to be fulfilled. As the optimal investment verifies

$$\partial_{q_k} H(a, k, \mathbf{p}_t, \partial_a V, \partial_k V) = i_t^* + g(k, \mathbf{p}_t),$$

this equilibrium equation can be written

$$\int_{\mathbb{R}^2} \partial_{q_k} H(a, k, \mathbf{p}_t, \partial_a V, \partial_k V)\rho_t(a, k)\mathrm{d}a\mathrm{d}k = \int_{\mathbb{R}^2} (g(k, \mathbf{p}_t) + \Xi(k, \mathbf{p}_t))\rho_t(a, k)\mathrm{d}a\mathrm{d}k.$$

As written in [5], solving this equation is equivalent as finding a fixed point to the function $\mathbf{p} \mapsto \tilde{\mathbf{p}}$ where $\tilde{\mathbf{p}}$ is calculated as the solution of the equilibrium equation in which V is calculated under the assumption that the price is $\mathbf{p}$.

3.1 Solution to the Transport Equation

The initial values $a_0^n = \mathbf{a}_0^n$ and $k_0^n = \mathbf{k}_0^n$ are given, so on the macroeconomic scale, the density of agents at the initial time $t = 0$ is given by

$$\rho_0 = \frac{1}{N} \sum_{n=1}^{N} \delta_{(a_0^n, k_0^n)},$$

where δ is the Dirac measure.

Proposition 3.1. *Suppose* $(\mathbf{a}^n, \mathbf{k}^n) \in C^0([0, T])^2$ *are given by assumptions 2.2 and 2.1. Then*

$$\rho_t := \frac{1}{N} \sum_{n=1}^{N} \delta_{(\mathbf{a}_t^n, \mathbf{k}_t^n)}$$

solves the transport equation 1.4 in the sense of distributions and verifies

$$\forall t \in [0, T], \int_{\mathbb{R}^2} \rho_t(\mathrm{d}x) = 1.$$

Proof

Let $t \in [0,T]$. By the definition of the Dirac measure, we have

$$\int_{\mathbb{R}^2} \rho_t = \frac{1}{N} \sum_{n=1}^{N} \int_{\mathbb{R}^2} \delta_{(\mathbf{a}_t^n, \mathbf{k}_t^n)}(\mathrm{d}x) = 1.$$

Let $\mathbf{v}(a,k,t) := \nabla_{q_a,q_k} H(a,k,\mathbf{p}_t, \partial_a V(a,k,t), \partial_k V(a,k,t))$ and $\nabla := \nabla_{a,k}$. The equation 1.4 can be written $\partial_t \rho + \mathrm{div}(\rho \mathbf{v}) = 0$. In the sense of distributions, for all $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^2)$, we have

$$\langle \mathrm{div}(\rho\mathbf{v}), \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle \rho, -\mathbf{v}\cdot\nabla\varphi\rangle = -\iiint_{[0,T]\times\mathbb{R}^2} \mathbf{v}(a,k,t)\cdot\nabla\varphi(a,k,t)\rho(\mathrm{d}a\mathrm{d}k\mathrm{d}t).$$

Assume $N = 1$, then

$$\begin{aligned}
\langle \mathrm{div}(\rho\mathbf{v}), \varphi \rangle &= \iiint_{[0,T]\times\mathbb{R}^2} \mathbf{v}(a,k,t)\cdot\nabla\varphi(a,k,t)\delta_{(\mathbf{a}_t,\mathbf{k}_t)}(\mathrm{d}a\mathrm{d}k\mathrm{d}t) \\
&= \int_0^T \mathbf{v}(\mathbf{a}_t^n, \mathbf{k}_t^n, t)\cdot\nabla\varphi(\mathbf{a}_t,\mathbf{k}_t,t)\mathrm{d}t \\
&= -\int_0^T (\dot{\mathbf{a}}_t, \dot{\mathbf{k}}_t)\cdot\nabla\varphi(\mathbf{a}_t,\mathbf{k}_t,t)\mathrm{d}t \\
&= \int_0^T \partial_t\varphi(\mathbf{a}_t,\mathbf{k}_t,t)\mathrm{d}t \\
&= -\langle \partial_t \delta_{(\mathbf{a}_t^n,\mathbf{k}_t^n)}, \varphi\rangle.
\end{aligned}$$

So we obtain $\langle \partial_t\rho + \mathrm{div}(\rho\mathbf{v}), \varphi\rangle = 0$. By linearity of the equation, the result is valid for all $N \in \mathbb{N}$. □

3.2 Computation of the Price

Define $\mathbf{q}_a^n(t) := \partial_a V(\mathbf{a}_t^n, \mathbf{k}_t^n, t)$ and $\mathbf{q}_k^n(t) := \partial_a V(\mathbf{a}_t^n, \mathbf{k}_t^n, t)$, where V is the value function defined by 1.1. As $V(\cdot,\cdot,T) = 0$, we obtain $\mathbf{q}_a^n(T) = 0$ and $\mathbf{q}_k^n(T) = 0$.

3.2.1 Computation of the Hamiltonian

Define $h(c,i,a,k,p,q_a,q_k) := (-c - pi + F(k,p))q_a + (i + g(k,p))q_k + u(c,i,a,k)$ and assume $u \in C^1(\mathbb{R}^4)$.

If there are no constraints on c, the Hamiltonian is given by

$$H(a,k,p,q_a,q_k) := \sup_{(c,i)\in\mathbb{R}^2} h(c,i,a,k,p,q_a,q_k).$$

For the optimal c and i, we have $\partial_c h = 0$ and $\partial_i h = 0$, what gives

$$\begin{cases} q_a &= \partial_c u(c^*, i^*, a, k) \\ q_k &= p\partial_c u(c^*, i^*, a, k) - \partial_i u(c^*, i^*, a, k) \end{cases} \tag{3.1}$$

If u is strictly concave in c and $\partial_c u(c,i,a,k) = u_c(c)$ depends only on c, then u_c is strictly decreasing and we can compute $c^* = u_c^{-1}(q_a)$. The equation for q_k can be written $\partial_i u(c^*, i^*, a, k) = pq_a - q_k$. As for c, if $\partial_i u(c,i,a,k) = u_i(i)$ depends only on i is strictly decreasing, then $i^* = u_i^{-1}(pq_a - q_k)$. Consequently, we obtain

$$H(a,k,p,q_a,q_k) = h(u_c^{-1}(q_a), u_i^{-1}(pq_a - q_k), a, k, p, q_a, q_k). \tag{3.2}$$

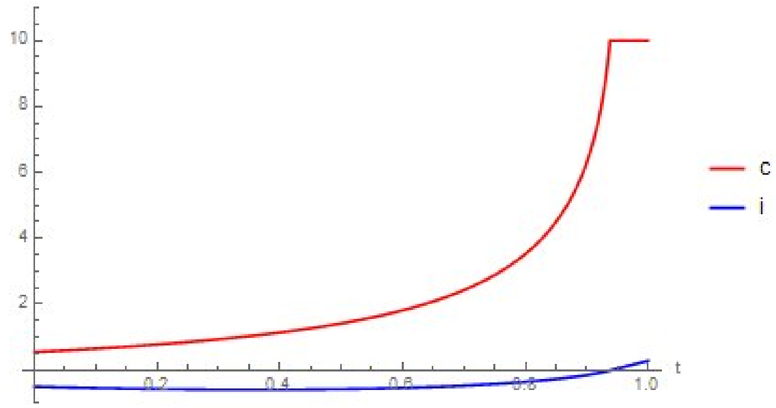

Figure 3.1: Consumption $\mathbf{c} < R$ and Investment $\mathbf{i}$ for $a_0 = k_0 = 1$ and $\mathbf{p}_t = 1 + t$.

If there exists $R \in \mathbb{R}_+^*$ such that $\mathbf{c} \leq R$, then

$$H(a, k, p, q_a, q_k) := \sup_{\substack{(c,i)\in\mathbb{R}^2 \\ c<R}} h(c, i, a, k, p, q_a, q_k).$$

For the optimal c and i, we can use a Lagrange multiplier $\lambda \leq 0$ and obtain

$$\begin{cases} \partial_c u(c^*, i^*, a, k) - q_a + \lambda &= 0 \\ \lambda(c^* - R) &= 0 \end{cases}$$

Thus, we have $c^* < R$ and $q_a = \partial_c u$ or $c^* = R$ and $q_a \leq \partial_c u$, what gives $c^* = \max(R, u_c^{-1}(q_a))$.

Note: If $u \in C^1(\mathbb{R}^4)$ is increasing and strictly concave in c, then $\partial_c u > 0$. If $\mathbf{q}_a$ is continuous, then $\mathbf{q}_a \underset{t\to T}{\to} 0$ and $\partial_c u(\mathbf{c}_t, \mathbf{i}_t, \mathbf{a}_t, \mathbf{k}_t) \underset{t\to T}{\to} 0$. Assume $\mathbf{c}$ is continuous. Then, if there is no constraint on c, we obtain

$$\mathbf{c}_t \underset{t\to T}{\to} +\infty,$$

and if there is the constraint $\mathbf{c} \leq R$, then there exists $t_0 \in [0, T[$ such that

$$\forall t \in [t_0, T], \mathbf{c}_t = R.$$

3.2.2 Computation of the Optimal Trajectories

Assume V is C^1. Then, as it is written in [5], V is the solution to the Hamilton-Jacobi equation 1.3 and we obtain the following Hamiltonian system

$$\begin{cases} \dot{\mathbf{a}}_t^n &= \partial_{q_a} H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) \\ \dot{\mathbf{k}}_t^n &= \partial_{q_k} H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}_a^n(t) &= -\partial_a H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}_k^n(t) &= -\partial_k H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) \end{cases} \tag{3.3}$$

what can be shown using the method of characteristics (see for example [3]). This system has initial/terminal conditions that are given by $\mathbf{a}_0^n = a_0^n$, $\mathbf{k}_0^n = k_0^n$, $\mathbf{q}_a^n(T) = 0$ and $\mathbf{q}_k^n(T) = 0$.

Reciprocally, the following proposition is proved in [5]:

Proposition 3.2. *Assume that the utility function $u \in C^1(\mathbb{R}^4)$ is concave and is non-decreasing in a and k, that the production function, F, is non-decreasing and concave in k and that the depreciation function, g, is concave in k.*
If $(\mathbf{a}, \mathbf{k}, \mathbf{q}_a, \mathbf{q}_k)$ is a solution to the system 3.3 with the given initial/terminal conditions, then $\mathbf{a}$ and $\mathbf{k}$ are optimal trajectories of the problem 2.1 and

$$\forall t \in [0, T], \mathbf{q}_a(t) \geq 0 \text{ and } \mathbf{q}_k(t) \geq 0$$

We can Compute the solutions to the Hamiltonian system 3.3 with initial conditions. It gives us a function $Q^n : (\mathbf{q}_a^n(0), \mathbf{q}_k^n(0)) \mapsto (\mathbf{q}_a^n(T), \mathbf{q}_k^n(T))$. Then, in order to find the optimal trajectories, we have to find $q \in \mathbb{R}^2$ such that $Q^n(q) = 0$, what can be done for example by minimizing $\|Q^n\|_2^2$.

3.2.3 Computation of the Price.

Assume we found for a given $\mathbf{p} : [0, T] \to \mathbb{R}$ the optimal trajectories of the N players, then we can find a new $\tilde{\mathbf{p}}$ verifying the equilibrium equation 1.5 by finding a solution to

$$\sum_{n=1}^{N} \partial_{q_k} H(\mathbf{a}_t^n, \mathbf{k}_t^n, \mathbf{p}_t, \mathbf{q}_a^n(t), \mathbf{q}_k^n(t)) = \sum_{n=1}^{N} g(\mathbf{k}_t^n, \mathbf{p}_t) + \Xi(\mathbf{k}_t^n, \mathbf{p}_t).$$

3.3 Numerical Results.

In all this section, we use $F(k,p) = k + 0.1kp$ and $g(k,p) = -|k|/2$.

3.3.1 c bounded by above and price $\mathbf{p} > 0$

In this section, we assume that c is bounded by above by $R = 10$. Let $u_1(x) := \left(\frac{1}{16} + x\right)^{1/2} \mathbb{1}_{\{x>0\}} + \left(\frac{5}{4} - (1-x)^2\right) \mathbb{1}_{\{x<=0\}}$. Then $u_1 \in C^1(\mathbb{R})$ and $u_1(x) < \frac{5}{4} + (x^+)^{1/2} - (x^-)^2$.

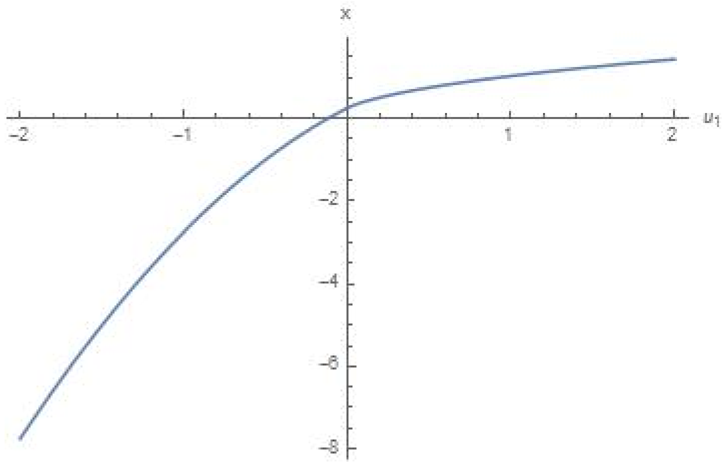

Figure 3.2: The Function u_1.

Let $u(c,i,a,k) := u_1(c) + u_1(a) + u_1(k) - |i|^2$. Then u verifies all the hypothesis of the Theorem 2.1. With the notations of previous section, we have $u_c = u_i = u_1'$ that are strictly decreasing, so we can compute $c^* = \max(R, u_c^{-1}(q_a))$ and then the Hamiltonian.

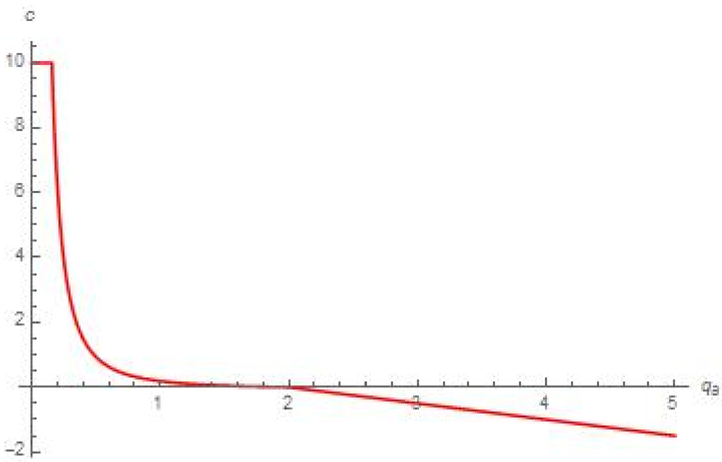

Figure 3.3: Consumption $c^* = \max(R, u_c^{-1}(q_a))$ as a Function of q_a.

Using the Hamiltonian system 3.3, we can compute, for a given price $\mathbf{p}$, the trajectories $(\mathbf{a}, \mathbf{k}, \mathbf{q}_a, \mathbf{q}_k)$ and also $\mathbf{c}_t = \max(R, u_c^{-1}(\mathbf{q}_a(t)))$ and $\mathbf{i}_t = u_i^{-1}(\mathbf{p}_t\mathbf{q}_a(t) - \mathbf{q}_k(t))$. Figure 3.3.1 shows the trajectories obtained for $a_0 = k = 0 = 1$ and $\mathbf{q}_a(0)$ ans $\mathbf{q}_k(0)$ chosen in order to obtain $\mathbf{q}_a(T) = \mathbf{q}_k(T) = 0$. In order to obtain the right initial conditions for $\mathbf{q}_a$ and $\mathbf{q}_k$, as written in the previous section, we minimize $\|Q(\mathbf{q}_a(0), \mathbf{q}_k(0))\|_2^2 = \mathbf{q}_a(T)^2 + \mathbf{q}_k(T)^2$. Figure 3.3.1 shows the function $\|Q\|_2^2$ near its local minimum in $[0,100]^2$. The algorithm to find the minimum calculates the function on a grid of 11×11 points and zooms recursively around the minimum until it reached a given precision. This minimum is assumed to be 0 as we reached $\|Q(0.639007, 0.896937)\|_2^2 = 2,74658.10^{-12}$ with a grid step of $5, 12.10^{-6}$.

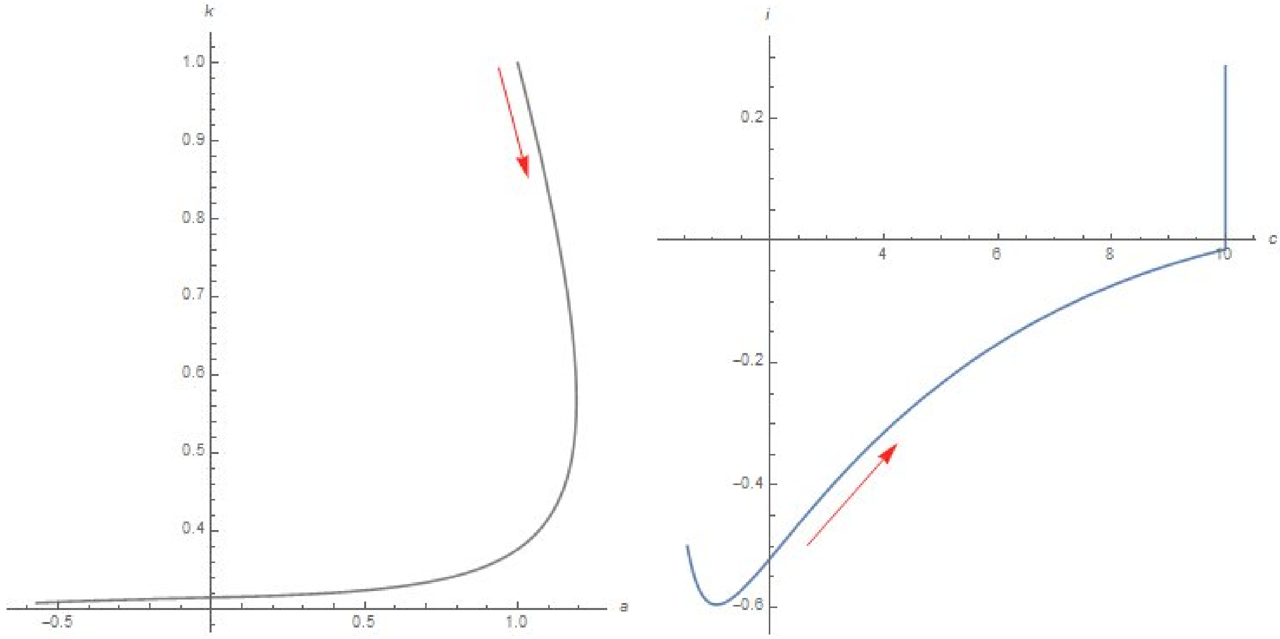

Figure 3.4: $(\mathbf{a}, \mathbf{k})$ and $(\mathbf{c}, \mathbf{i})$ Trajectories for $t < 1$, $a_0 = k_0 = 1$ and $\mathbf{p}_t = 1 + t$.

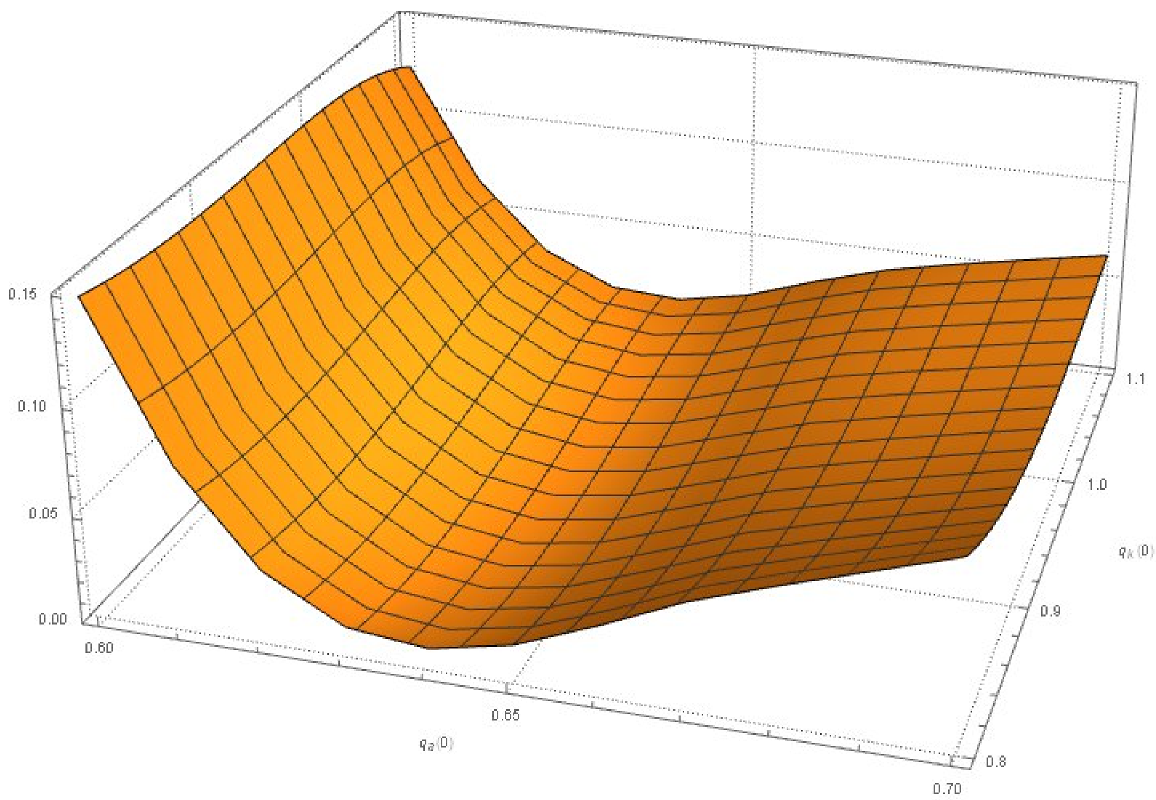

Figure 3.5: $\|Q(\mathbf{q}_a(0), \mathbf{q}_k(0))\|_2^2$ for $T = 1$, $a_0 = k_0 = 1$ and $\mathbf{p}_t = 1 + t$.

3.3.2 Price $\mathbf{p} = 0$.

In the case of $\mathbf{p} = 0$, the minimum of $\|Q\|_2^2$ is strictly positive for our settings. As written in previous section, the system 3.3 is valid if V is C^1 and if this system has a solution, then this solution is the good one, but it is not sure that this system have a solution if V is not C^1. It means that V is not C^1 in $a = k = 1$. Figure 3.3.2 shows the function $\|Q\|_2^2$ near its minimum. This minimum is strictly above 0.02.

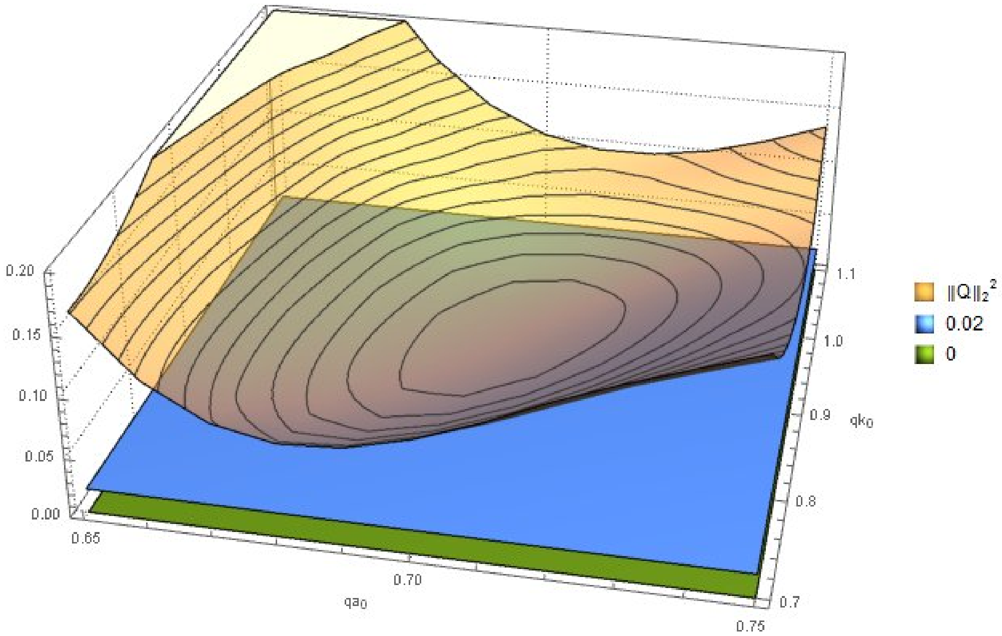

Figure 3.6: $\|Q(\mathbf{q}_a(0), \mathbf{q}_k(0))\|_2^2$ for $T = 1$, $a_0 = k_0 = 1$ and $\mathbf{p}_t = 0$.

As it is written in [5], it is possible to show the uniqueness of a solution to our maximization problem 2.1 and then that $V \in C^1(\mathbb{R}^3)$ with the assumption of uniform convexity on the function u. It is clearly not the case here, as u is increasing in c, a and k.

3.3.3 Utility C^2 and Stationary in the c variable

The fact that we have $c \leq R$ could be equivalently written $u = -\infty$ when $c > R$, or $u(c, i, a, k) = u(R, i, a, k)$ for $c \geq R$, what means that u is no more continuous, or at least no more C^1 as u is strictly increasing in the c variable. In order to prevent this, it is possible to take u as a C^2 function, constant for $c \geq R$.

Let $u_2(x) := \left(\frac{x}{R} - 1\right)^3 \mathbb{1}_{\{x<R\}}$. Then $u_2 \in C^2(\mathbb{R})$ and $u_2(x) < R^{-3}((x^+)^0 - (x^-)^3)$.

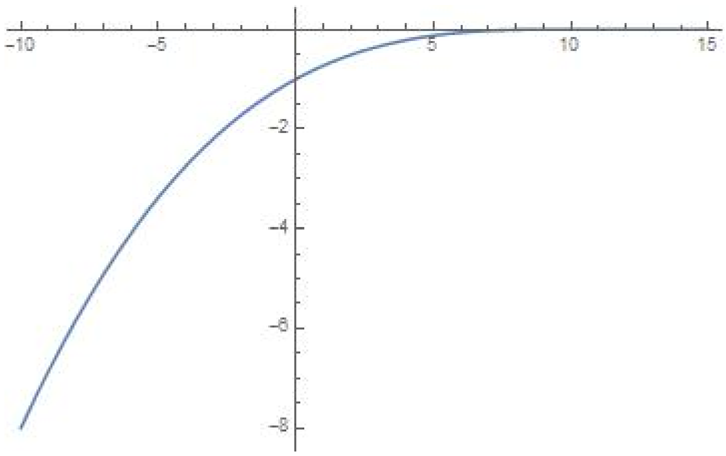

Figure 3.7: The Function u_2.

Let $u(c, i, a, k) := u_2(c) + u_1(a) + u_1(k) - |i|^2$. Then u verifies all the hypothesis of the Theorem 2.1. But u is no more strictly convex. So, as uniform convexity is stronger than strict convexity, we are even more far from the hypothesis that are used in [5] to show the uniqueness of a solution to our problem. Figure 3.3.3 shows that with this utility function, for $\mathbf{p}_t = 1 + t$ and $a_0 = k_0 = 1$, there are two optimal trajectories.

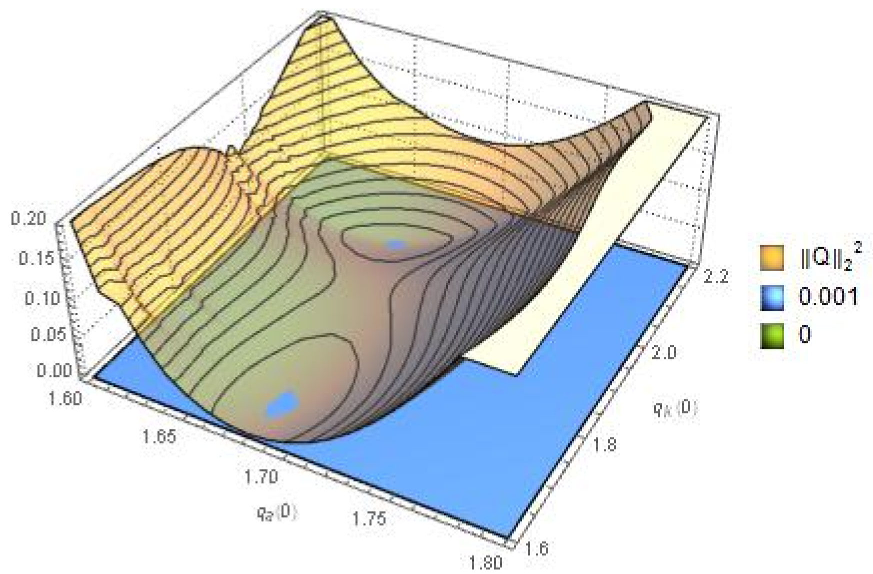

Figure 3.8: $\|Q(\mathbf{q}_a(0), \mathbf{q}_k(0))\|_2^2$ for $T = 1$, $a_0 = k_0 = 1$ and $\mathbf{p}_t = 1 + t$

Conclusion and Personal Experience

This internship allowed me to discover the world of research in applied mathematics. I first learned a lot in topic related to functional analysis, such as partial differential equations, optimal control and calculus of variation, and more specifically about mean field games theory, through seminars and courses from my supervisor, members of my group, and other researchers from KAUST or coming from other universities. My work was done in parallel with the creation of the book *Economic Models and Mean-Field Games Theory* [5] by members of my group. I focused on the problem of the conditions for the existence of trajectories in the mean field game exposed in this report. The first step was perhaps the most difficult but also the most interesting. It consisted to find the approach and the reasonable hypothesis on the model, even more difficult as there are several formulations of the problem. I first tried several formulations and tried several more simple examples to better understand the problem. My supervisor helped me a lot by bringing to me all the theoretical material that could be used and by advising me to work with growth conditions on the utility. The second step consisted in writing the proof, first in a very informal way with some precise parts, and then in the final and precise version, that forced me to work more on the definitions and the spaces used in the proofs. In the last part of the internship, I tested several configurations on concrete numerical examples using Mathematica.

The economic model I studied was a rather simple model, but it still explains several behaviours and brings interrogations that can arise from the economical point of view as well as from the daily point of view. The book [5] also considers models with maximization of the utility on an infinite time period with a discount rate associated to the utility, and models that have a final cost. The current model suffers from the fact that there is no final cost, what creates this effect of maximal consumption at the end of the maximization period.

The numerical tests showed that all the formulations of the problem are not valid at the same time. The hypothesis of a strictly positive price seems to have a role in the existence of a continuously differentiable value function, what would bring reliable algorithms to compute the mean field game solution, but this hypothesis is not required to have existence of the value function. Moreover, a study of better algorithms and the convergence of algorithms to find the optimal trajectories and the price could be continued. This work does indeed not show any properties on the price unless the fact that it should be almost everywhere bounded, what makes it difficult to find the good class of functions that should be used to approximate it, and does not show that the fixed point algorithm used to find the price always converge. The best option to remedy to these problems could be to study more detailed models.

Bibliography

[1] H. Brezis. *Analyse fonctionnelle : Théorie et applications*. Sciences Sup. Dunod, Paris, nouvelle présentation 2005 edition, 2005.

[2] P. Cardaliaguet. Notes on mean field games. Technical report, Technical report, 2010.

[3] L. C. Evans. *Partial Differential Equations: Second Edition*. Number 19 in Graduate studies in mathematics. American Mathematical Society, Providence, R.I, 2nd edition, Mar. 2010.

[4] D. A. Gomes, J. Mohr, and R. R. Souza. Continuous Time Finite State Mean Field Games. *Applied Mathematics & Optimization*, 68(1):99–143, Aug. 2013.

[5] D. A. Gomes, L. Nurbekyan, and E. A. Pimentel. *Economic Models and Mean-field Games Theory*. Number 4 in Colóquio Brasileiro de Matemática 30º - 2015. IMPA, Brazil, 1st edition, Aug. 2015.

[6] D. A. Gomes and J. Saúde. Mean Field Games Models—A Brief Survey. *Dynamic Games and Applications*, 4(2):110–154, June 2014.

[7] T. H. Gronwall. Note on the Derivatives with Respect to a Parameter of the Solutions of a System of Differential Equations. *Annals of Mathematics. Second Series*, 20(4):292–296, July 1919.

[8] O. Guéant, J.-M. Lasry, and P.-L. Lions. Mean Field Games and Applications. In *Paris-Princeton Lectures on Mathematical Finance 2010*, number 2003 in Lecture Notes in Mathematics, pages 205–266. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[9] M. Huang, P. Caines, and R. Malhame. Large-Population Cost-Coupled LQG Problems With Nonuniform Agents: Individual-Mass Behavior and Decentralized Epsilon-Nash Equilibria. *IEEE Transactions on Automatic Control*, 52(9):1560–1571, Sept. 2007.

[10] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[11] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I – Le cas stationnaire. *Comptes Rendus Mathematique*, 343(9):619–625, Nov. 2006.

[12] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II – Horizon fini et contrôle optimal. *Comptes Rendus Mathematique*, 343(10):679–684, Nov. 2006.

[13] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, Mar. 2007.

[14] P.-L. Lions. Jeux à champ moyen, 2006.

Appendix A

Gronwall Type Inequalities

Proposition A.1 (Gronwall's Inequality - differential form [7]). *Let $I \subset \mathbb{R}$ be an open interval. Let $(a,b,u) \in C^0(I)^3$ such that u is differentiable in $\overset{\circ}{I}$ and that*

$$\forall t \in \overset{\circ}{I}, u'(t) \leq a(t)u(t) + b(t)$$

Then defining $A(t) := \int_0^t a$,

$$\forall t \in I, u(t) \leq e^{A(t)} \left(u(0) + \int_0^t b(s) e^{-A(s)} \mathrm{d}s \right)$$

And if $a \geq 0$ and $b \geq 0$ then

$$\forall t \in I, u(t) \leq e^{A(t)} \left(u(0) + \int_0^t b \right)$$

Proposition A.2 (Gronwall's Inequality - almost everywhere [3]). *Let $I \subset \mathbb{R}$ be an open interval.*
Let $(a,b,u) \in L^1(I)^3$ such that u is absolutely continuous and that

$$u'(t) \leq a(t)u(t) + b(t) \text{ a.e. on } I.$$

Then defining $A(t) := \int_0^t a$,

$$\forall t \in I, u(t) \leq e^{A(t)} \left(u(0) + \int_0^t b(s) e^{-A(s)} \mathrm{d}s \right)$$

And if $a \geq 0$ and $b \geq 0$ then

$$\forall t \in I, u(t) \leq e^{A(t)} \left(u(0) + \int_0^t b \right)$$

Proposition A.3 (Gronwall's Inequality - integral form). *Let $I \subset \mathbb{R}$ be an open interval. Let $(a,b,u) \in L^1_{loc}(I)^3$ such that*

$$\forall t \in I, u(t) \leq b(t) + a(t) \int_0^t u$$

Then

$$\forall t \in I, u(t) \leq b(t) + a(t) e^{A(t)} \int_0^t b(s) e^{-A(s)} \mathrm{d}s$$

And if $a \geq 0$ and $b \geq 0$ then

$$\forall t \in I, u(t) \leq b(t) + a(t) e^{A(t)} \int_0^t b$$

Appendix B

Cauchy-Lipschitz Theorem Variations

For the following theorems, let $d \in \mathbb{N}\backslash\{0\}$, $I \subset \mathbb{R}$ be an open interval, Ω be an open subset of $\mathbb{R}^d$ and $(x_0, t_0) \in \Omega \times I$.

Lemma B.0.1. *Let $F : \Omega \times I \to \mathbb{R}^d$ be locally uniformly integrable in the second variable. Then for all compact subsets $X \subset \Omega$ and $J \subset I$,*

$$\left(t \mapsto \sup_{x\in X} F(x,t)\right) \in L^1(J).$$

Proof

It follows from the fact that we can extract a finite subcover of open subsets of $X \times J$ such that the assertion holds for each open subset. □

Theorem B.1 (Cauchy-Lipschitz)**.** *(see for example [1])*
Let $F \in C^0(\Omega \times I, \mathbb{R}^d)$ be locally uniformly Lipschitz in the first variable.
Then there exists $J \subset I$ a neighbourhood of t_0 such that

$$\exists! \mathbf{x} \in C^0(J, \Omega), \mathbf{x}(t_0) = x_0 \text{ and } \forall x \in J, \dot{\mathbf{x}}(t) = F(\mathbf{x}(t), t)$$

Theorem B.2 (Cauchy-Lipschitz - locally integrable in t)**.**
Let $F : \Omega \times I \to \mathbb{R}^d$ be locally uniformly Lipschitz in the first variable and locally uniformly integrable in the second variable.
Then there exists $J \subset I$ a neighbourhood of t_0 such that

$$\exists! \mathbf{x} \in C^0(J, \Omega), \mathbf{x}(t_0) = x_0 \text{ and } \dot{\mathbf{x}}(t) = F(\mathbf{x}(t), t) \text{ in } \mathcal{D}'(J)$$

Moreover, the solution is absolutely continuous.

Proof

As F is locally uniformly Lipschitz in the first variable, let $L \in \mathbb{R}_+^*$ and $(\delta x, \delta t_0) \in (\mathbb{R}_+^*)^2$ such that $\forall (x_1, x_2, t) \in \bar{B}(x_0, \delta x)^2 \times \bar{B}(t_0, \delta t_0), \|F(x_1, t) - F(x_2, t)\| \le L\|x_1 - x_2\|$. $X := \bar{B}(x_0, \delta x)$ is a compact because $X \subset \mathbb{R}^d$ that is a finite-dimensional space. Let $t \in \bar{B}(t_0, \delta t_0)$. $F(\cdot, t) \in C^0(X)$ because it is Lipschitz, consequently it is bounded on X and we can define

$$M := \begin{pmatrix} \bar{B}(t_0, \delta t_0) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sup_{x\in X} \|F(x,t)\| \end{pmatrix}.$$

By Lemma B.0.1, $M \in L^1(\bar{B}(x_0, \delta t_0))$, and as $\int_{\bar{B}(t_0,r)} M \underset{r\to 0}{\to} 0$, there exists $\delta t_1 \in]0, \delta t_0]$ such that $\int_{\bar{B}(t_0,\delta t_1)} M < \delta x$. Let $\delta t := \min(\delta t_1, \frac{1}{2L})$, $J := \bar{B}(t_0, \delta t)$, $\mathcal{C} := C^0(J, X)$ and

$$\Phi := \begin{pmatrix} \mathcal{C} & \longrightarrow & \mathcal{C} \\ \mathbf{x} & \longmapsto & t \mapsto x_0 + \int_{t_0}^t F(\mathbf{x}(s), s)\mathrm{d}s \end{pmatrix}.$$

For $\mathbf{x} \in \mathcal{C}, \Phi(\mathbf{x})$ is continuous because $\int_J \|F(\mathbf{x}(s), s)\| \mathrm{d}s \leq \int_J M$ and for $t \in J$, as $\delta t \leq \delta t_1$, we have $\|\Phi(\mathbf{x})(t) - x_0\| \leq \int_J M \leq \delta x$. Thus, we have $\Phi(\mathcal{C}) \subset \mathcal{C}$ and $\boldsymbol{\Phi}$ **is well defined**.

Let $(\mathbf{x}, \bar{\mathbf{x}}) \in \mathcal{C}^2$, then

$$\begin{aligned} \|\Phi(\mathbf{x}) - \Phi(\bar{\mathbf{x}})\|_\infty &\leq \int_{t_0}^{t} \|F(\mathbf{x}(s), s) - F(\bar{\mathbf{x}}(s), s)\| \mathrm{d}s \\ &\leq L \int_{t_0}^{t} \|\mathbf{x}(s) - \bar{\mathbf{x}}(s)\| \mathrm{d}s \\ &\leq L\delta t \, \|\mathbf{x} - \bar{\mathbf{x}}\|_\infty, \end{aligned} \tag{B.1}$$

where $L\delta t < 2L\delta t \leq 1$. Thus, as $(\mathcal{C}, \|\cdot\|_\infty)$ is a Banach space, by Banach fixed-point theorem, there exists a unique $\mathbf{x} \in \mathcal{C}$ such that $\Phi(\mathbf{x}) = \mathbf{x}$.

By Lebesgue differentiation theorem, we can differentiate $\Phi(\mathbf{x})$ a.e., thus we have

$$\text{for a.e. } t \in J, \ \dot{\mathbf{x}}(t) = F(\mathbf{x}(t), t).$$

By the absolute continuity of the Lebesgue integral, we obtain that $\mathbf{x}$ is absolutely continuous and that its derivative in the sense of distributions verifies $\dot{\mathbf{x}} \in L^1(J)$. As F is locally uniformly integrable in the second variable, $t \mapsto F(\mathbf{x}(t), t)$ is also in $L^1(J)$. So we obtain $\dot{\mathbf{x}} = F(\mathbf{x}(\cdot), \cdot)$ in $L^1(J)$, what gives

$$\dot{\mathbf{x}} = F(\mathbf{x}(\cdot), \cdot) \text{ in } \mathcal{D}'(J)$$

As $\mathbf{x} \in \mathcal{C} \subset C^0(J, \Omega)$, it shows the **existence of the solution in** $C^0(J, \Omega)$.

Assume there exists also $\bar{\mathbf{x}} \in C^0(J, \Omega) \backslash \mathcal{C}$ verifying the same Cauchy problem. Then there exists $t_2 \in J$ such that $\bar{\mathbf{x}}(t_2) \notin X$. Without loss of generality, we can assume that $t_2 > t_0$. As $\bar{\mathbf{x}}$ is continuous and $\bar{\mathbf{x}}(t_0) \in X$, we obtain that $t_1 := \max\{t \in [t_0, t_2], \bar{\mathbf{x}}(t) \in X\} \in]t_0, t_2[$ and $\|\bar{\mathbf{x}}(t_1) - \bar{\mathbf{x}}(t_0)\| = \delta x$. But as $\mathbf{x}$ is absolutely continuous and $\dot{\mathbf{x}} = F(\mathbf{x}(\cdot), \cdot)$, we also have

$$\|\bar{\mathbf{x}}(t_1) - \bar{\mathbf{x}}(t_0)\| \leq \int_{t_0}^{t} \|F(\bar{\mathbf{x}}(s), s)\| \mathrm{d}s \leq \int_J M < \delta x,$$

which is a contradiction. Thus, **there exists a unique solution in** $C^0(J, \Omega)$. □

Corollary B.1.
Let F be defined as in previous theorem and $f \in L^1_{loc}(I)$.
Then there exists $J \subset I$ a neighbourhood of t_0 such that

$$\exists! \mathbf{x} \in C^0(J, \Omega), \mathbf{x}(t_0) = x_0 \text{ and } \forall x \in J, \dot{\mathbf{x}}(t) = f(t) + F(\mathbf{x}(t), t)$$

Theorem B.3. *Let P be a Cauchy problem defined for $t \in I$ where $I \subset \mathbb{R}$ is an open interval. Let $x : J \mapsto \Omega$ be the maximal solution to P. If $\sup J < \sup I$ then*

$$\forall X \subset \Omega, X \text{ compact} \Rightarrow \exists \tau \in J, \forall t > \tau, x(t) \notin X.$$

Appendix C

Inequalities and Concave functions

C.1 Some Inequalities

The following inequalities are an easy way to express the intuitive fact that on finite time integrals, only the highest exponent has to be taken into account.

Lemma C.1.1. *Let* $(p,q) \in (\mathbb{R}^+)^2$*. Then* $\forall x \in \mathbb{R}^+, x^p + x^q \leq 1 + 2x^{\max(p,q)}$

Proof

Let $(p,q) \in (\mathbb{R}^+)^2$ and $x \in \mathbb{R}^+$. Suppose $p \leq q$.
If $x \leq 1$ then $x^p \leq 1$ thus $x^p + x^q \leq 1 + x^q \leq 1 + 2x^q$ $\square$

Lemma C.1.2. *Let* $\gamma \in [0,1]$ *and* $\alpha \in [1,+\infty[$*. Then* $\forall x \in \mathbb{R}^+, x^\gamma \leq 1 + x^\alpha$

Lemma C.1.3. *Let* $\gamma \in [0,1]$ *and* $\alpha \in [1,+\infty[$*. Then* $\forall M \in \mathbb{R}_+, \forall x \in \mathbb{R}^+, x^\gamma \leq M^\gamma + \dfrac{x^\alpha}{M^{\alpha-\gamma}}$

C.2 Concave Functions

Proposition C.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a concave function. Then* f *is locally Lipschitz.*

Proof

Let $(a,b) \in \mathbb{R}^2$ such that $a < b$ and let define $d_a := f(a) - f(a-1)$ and $d_b := f(b+1) - f(b)$.
As f is concave, then for $(x,y) \in [a,b]^2$ such that $x < y$,

$$d_b \leq \frac{f(b) - f(y)}{b - y} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(x) - f(a)}{x - a} \leq d_a$$

thus we have $|f(y) - f(x)| \leq \max(|d_a|, |d_b|)|y - x|$ $\square$

Proposition C.2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $\forall y \in \mathbb{R}, f(\cdot, y)$ *is a concave function, and that* $\forall x \in \mathbb{R}, f(x, \cdot)$ *is locally bounded. Then* $f \in \mathcal{L}^\infty_{loc}(\mathbb{R}^2)$*.*

Proof

Let $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$ such that $x_1 < x_2$, $y_1 < y_2$ and $\forall x \in [x_1-1, x_2+1] \times [y_1, y_2], f(x,y) \leq M(x)$.

As in the previous demonstration, we can define for all $y \in [y_1, y_2], d_1(y) := f(x_1, y) - f(x_1 - 1, y)$ and $d_2(y) := f(x_2 + 1, y) - f(x_2, y)$ and $f(\cdot, y)$ is $\max(|d_1(y)|, |d_2(y)|)$-Lipschitz

in $[x_1, x_2]$.
Then we have for $y \in [y_1, y_2]$:

$$\begin{aligned} |d_1(y)| &= |f(x_1, y) - f(x_1 - 1, y)| \\ &\leq |f(x_1, y) - f(x_1, y_1)| + |f(x_1, y_1) - f(x_1 - 1, y_1)| + |f(x_1 - 1, y_1) - f(x_1 - 1, y)|) \\ &\leq M(x_1) + |d_1(y_1)| + M(x_1 - 1) \\ |d_2(y)| &\leq M(x_2) + |d_2(y_1)| + M(x_2 - 1) \end{aligned}$$

Thus for all y in $[y_1, y_2]$, $f(\cdot, y)$ is L-Lipschitz with $L = \max(M(x_1) + |d_1(y_1)| + M(x_1 - 1), M(x_2) + |d_2(y_1)| + M(x_2 - 1))$.

Let $(x, y) \in [x_1, x_2] \times [y_1, y_2]$.

$$\begin{aligned} |f(x, y) - f(x_1, y_1)| &\leq |f(x, y) - f(x_1, y)| + |f(x_1, y) - f(x_1, y_1)| \\ &\leq L|x_2 - x_1| + 2\|f(x_1, \cdot)\|_{L^\infty(y_1, y_2)} \end{aligned}$$

Thus $\|f\|_{\mathcal{L}^\infty([x_1,x_2]\times[y_1,y_2])} \leq |f(x_1, y_1)| + L|x_2 - x_1| + 2\|f(x_1, \cdot)\|_{L^\infty(y_1,y_2)}$ □